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Developing Secondary School Teachers' Didactic–Mathematical Knowledge about Probability

Desenvolvendo o Conhecimento Didático-Matemático de Professores do Ensino Secundário sobre Probabilidade

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Abstract

A formative experience oriented towards development of didactic-mathematical knowledge on probability and associated notions with mathematics teachers working of Fundamental Education in Brazil is described. Theoretical tools from the "onto-semiotic approach of mathematical knowledge and instruction" are used to design and analyze the formative experience. The phases of a didactic design based on this theoretical framework are shown composing the main thread of the developed experience. In the same way, the categories of common, advanced and specialized knowledge, from the mathematics teacher's didactic-mathematical knowledge model, are applied. The formative model designed, especially the sequence of proposed activities and their a priori analysis, is a contribution allowing to support and educate adequately mathematics teachers on the specific issue of probability and its didactic.

Keywords: Teacher Education. Probability. Didactic-Mathematical Knowledge. Onto-Semiotic Approach.

Resumo

Descreve-se uma experiência formativa orientada para o desenvolvimento do conhecimento didático-matemático sobre probabilidade e noções associadas com professores de matemática que atuam no Ensino Fundamental no Brasil. Ferramentas teóricas da "abordagem ontossemiótica do conhecimento e instrução matemática" são usadas para o desenho e analise sa experiência formativa. As fases de um desenho didático baseado neste referencial teórico são apresentadas como o fio condutor da experiência desenvolvida. Da mesma forma, as categorias de conhecimento comum, avançado e especializado, do modelo de conhecimento didático-matemático do professor de matemática, são aplicadas. O modelo formativo desenhado, especialmente a sequência das atividades propostas e sua análise a priori, é uma contribuição que permite apoiar e formar adequadamente os professores de matemática sobre a questão específica da probabilidade e sua didática.

Palavras-chave*: Formação Docente. Probabilidade. Conhecimento Didático-Matemático. Abordagem Ontossemiótica.*

1 Introduction

In this article we describe a formative experience related to development of the Didactic-Mathematical Knowledge (DMK) of a group of mathematics teachers of the last years of primary school (students from 11 to 14 years of age) who participated in the education program of Anhanguera University of São Paulo's Observatory of Education in Brazil, in cooperation with the São Paulo State Secretariat of Education.

The preliminary analysis, design, implementation and evaluation of the activities of this education program were based on the Onto-Semiotic Approach (OSA) to Mathematical Knowledge and Instruction (Godino, Batanero & Font, 2007) and the theory of Didactic-Mathematical Knowledge (DMK) (Godino, 2009).This theoretical framework endeavors to express and bring together theories in mathematical education research based on anthropological and semiotic assumptions on the mathematics activity and their respective teaching processes. In our case, this theoretical framework helps us to question the mathematical skill on which the research is centered (basic probabilistic concepts) and the Didactic-Mathematical Knowledge of this skill that the teachers should put into practice in the teaching and learning processes.

This work is part of a research project whose objectives are:

• To prepare an education program for teachers of the last years of primary school on probability, beginning with a discussion of the concepts of randomness and sample space and, subsequently, the quantification of probability followed by the concept of risk.

• To investigate to what extent this program promotes the building of Didactic-Mathematical Knowledge of probability for teachers in the last years of primary school.

Various researchers have established the importance of probability teaching and learning (Batanero, Henry and Parzysz, 2005; Gal, 2005) in basic education. However, the consensus of research in this area (Ives, 2009; Batanero and Díaz, 2012) is that the topic of probability in primary and high school, whenever addressed, is restricted to procedural manipulation of formulas, and that teachers, even those of mathematics, deviate from probabilistic reasoning . Other researchers (Pietropaolo, Campos, Felisberto de Carvalho and Teixeira, 2013; Kataoka et al., 2008) studying probability within the Brazilian teaching scenario, suggest that although teachers usually cover probability and statistics at the undergraduate level, they do not develop specialized knowledge for teaching these concepts in professional practice. An original aspect of our investigation is the focus, in the formative process, on the articulation of mathematical content and its teaching. This enabled teachers to increase their knowledge about probability as they discussed didactic issues related teaching this topic. The theoretical framework adopted for this design and analysis of the formative intervention serves to support definition of this relationship. The concept of didactic engineering based on OSA, involving the phases of preliminary study, design, implementation and evaluation (Godino, Rivas, Arteaga, Lasa and Wilhelmi, 2014) is also used as a guideline for this formative intervention.

In the following section, we summarize the investigative subject and the theoretical framework by providing a brief description of the Onto-Semiotic Approach to Mathematical Knowledge and Instruction (OSA), as well as of the teacher knowledge categories and the phases of didactic engineering underlying this theory. In Section 3, we present the methodological route we followed and discuss the selection and adaptation of the education program activities. In Section 4, we analyze and illustrate the activities included in this formative program to promote development of basic and advanced knowledge of probability content. Subsequently, in Section 5, we provide some examples of activities to develop Specialized Content Knowledge (SCK).Finally, in Section 6, we present the final considerations and the implications for educating mathematics teachers on probability.

2 Problem and Theoretical Framework

The knowledge we emphasize in this work – that teachers should have to adequately teach mathematics, as stated by Godino, Batanero, Rivas and Arteaga (2013, p.71) – "implies a connection between mathematics and didactics". There are various theoretical models that reflect on the knowledge that teachers should put into practice to promote student learning. Shulman (1987) identified a special domain of teacher knowledge, which he named Pedagogical Content Knowledge (PCK). This author recommends different categories to analyze teacher's professional knowledge in a wider context than just teaching. Hill, Ball and Schilling (2008) suggest a refinement of the categories proposed by Shulman (1987) posing, among other questions: What do teachers need to know and what are they effectively capable of doing to develop the work of teaching mathematics? Hill et al. (2008) develop the idea of Mathematical Knowledge for Teaching (MKT) distinguishing six main categories for this concept, organized in Mathematical Content Knowledge (MCK) and Pedagogical Content Knowledge (PCK). These models support numerous works for initial and continuing teacher education.

The model of teacher's Didactic-Mathematical Knowledge (DMK), based on the OSA framework, proposes tools for discussing, identifying and classifying knowledge required for teaching mathematics, making it useful not only for the classroom study process, but also for teacher education programs. This model is an expansion of the concepts already discussed in the MKT model (Hill et al., 2008).The DMK model presents four different levels or points of view that provide additional categories of teacher knowledge. A description of these levels follows (Godino, Ortiz, Roa and Wilhelmi, 2011):

a) Mathematical-statistical and didactical practices: mathematical or statistical actions taken by students to solve problems, as well as actions taken by the teacher in order to promote learning and provide context for the content.

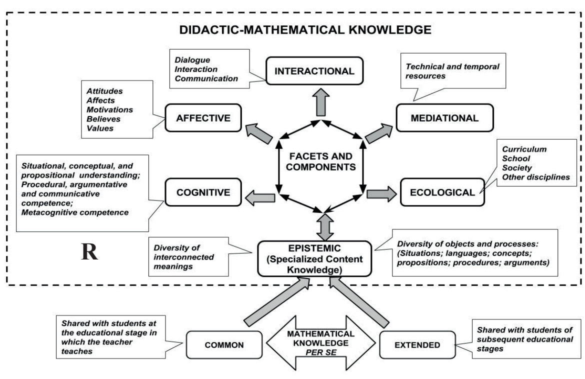
b) Configurations of objects and mathematical or statistical processes: mathematical objects (problems, procedures, concepts, properties, language and arguments) and processes (for example, generalization, representation) that intervene in and emerge from the aforementioned practices.

c) Norms: rules, habits and conventions that condition and make possible the study process and affect each facet and its interactions.

d) Didactical suitability: objective criteria that serve to improve teaching and learning, as well as to guide evaluation of teaching and learning processes.

This model also includes six facets involved in the teaching and learning processes for specific mathematical content. Figure 1 below shows the categories of teacher knowledge according to the Didactic Mathematical Knowledge (DMK) model.

Figure 1 - Facets and components of Didactic-Mathematical Knowledge (DMK Model)



Source: Godino, Batanero, Font y Giacomone (2016, p. 292)

Although the components are separate in Figure 1 in order to emphasize their differences, in reality, they all interact with each other. We will briefly discuss each category and its respective characteristics.

Common Content Knowledge (CCK) is the knowledge shared with students at the educational stage in which the teacher teaches. In our formative design, the activities were selected and adapted considering the probabilistic content necessary for the teachers in the final years of primary education.

Advanced Content Knowledge (ACK) is knowledge shared with the students in the subsequent educational stage. The teacher should have good mastery of the probabilistic concepts and deep understanding of them to organize teaching and put it into practice. We included activities dealing with conditional probability, different probability distributions and the normal curve, which in the Brazilian curriculum is dedicated to the secondary school level.

Specialized Content Knowledge (SCK) is a type of specific teacher knowledge (didactic knowledge) that considers the facets presented in Figure 1, such as the diversity of content meanings and the corresponding configurations of related objects and processes. These facets will be discussed below. The activities in our formative design addressing this category were selected in a way to allow teachers to make deep observations and/or prepare justifications and arguments related to probability. For example, to understand the importance of "first addressing the concept of chance and randomness before working on quantification of probabilities". Moreover it helps teachers propose useful mathematical activities and know how to deal with student's errors.

We present the facets of Specialized Content Knowledge (SCK) below, which in our case are oriented to the study of probability and the ideas that support this concept:

Epistemic facet: Intended and implemented institutional meaning for a given mathematical content (problems, procedures, concepts, properties, language, arguments) and its different meanings.. We believe that it is more complex for some teachers to acquire this component of their probabilistic knowledge rather than other mathematical content, due to the inherent complexity of randomness and the lack of internal epistemological uniqueness in which five valid institutional probability meanings are recognized, as noted by Batanero, Henry and Parzysz (2005), namely: Intuitive, classic, frequentist, subjective and axiomatic.

Cognitive facet: Students' levels of development and understanding, strategies, difficulties and errors as regards the intended content (personal meaning). For example, to understand the ways of thinking, difficulties and personal significances that students may present when working with probability.

Affective facet: knowledge of the degree of implication (interest or motivation) of students in the study process, their feelings and all the emotional components (attitudes, emotions, beliefs).

Mediational facet: knowledge of how to use appropriate didactic resources of all types (books, texts, technological or manipulative resources) for each theme and of the adequate time distribution for developing the teaching and learning processes, according to the educational level or degree the teaching is designed for.

Interactional facet: Organization of the classroom discourse and the interactions between the teacher and students directed to solve students' difficulties and negotiation of meanings.

Ecological facet: Relationships of the topic with other topics and with the social, political and economic settings that support and condition the teaching and learning of mathematics.

The didactical suitability of an instructional process is defined as the degree to which that process (or part thereof) fulfills certain characteristics being classified as suitable (optimal or appropriate) to get the adaptation between the personal meanings achieved by students (learning) and the intended or implemented institutional meanings (teaching), taking into account the circumstances and available resources (environment).Didactic suitability should be appraised for each of the six facets described above, since the study process can be suitable from the statistical point of view, but not suitable, for example, from the affective point of view. Consequently, six different types of suitability can be considered connected in a coherent and systemic manner (Godino, 2009), namely: epistemic, cognitive, affective, mediational, interactional and ecological suitability. However, these facets should not be considered independently –they relate to each other.

In the next sections we apply this theoretical system to analyze a formative intervention aimed to develop teacher's Didactic-Mathematical Knowledge on probability.

The examples herein discussed are, together with other activities, part of the preliminary and design study phases, distinguished by the idea of didactic engineering based on OSA (Godino, et al., 2014).

3 Method

The research was conducted in the context of the Education Observatory Project of São Paulo Anhanguera University , financed by the Coordination for the Improvement of Higher Education Personnel (CAPES) in cooperation with the São Paulo State Secretariat of Education. The proposal of the Education Observatory, according to Pietropaolo, Campos and Silva (2012),

> [...] is to establish a collaborative formative and research group whose purpose is to promote and analyze the professional development of mathematics teachers when involved in processes of implementing curricular innovations and to reflect on teaching practices (p. 379).

The education program involves 40 teachers, of which 23 (57%) are women and 17 (42.5%) are men. They met seven times for approximately four hours every two-week period. It should be emphasized that the participation of the teachers was spontaneous, however they were awarded with certificates from the São Paulo State Secretariat of Education. At the end of each study unit, the teachers received a guide for the activities proposed with a brief theoretical discussion of teaching and learning probability. Audiovisual record of the meetings was made and the written solutions and teachers' reflections were recorded for some activities, as well as the initial diagnosis. Research approved by the Brazilian Ethics Committee - number 645.337 of 12.05.2014.

Didactic engineering based on OSA was used to structure the phases of our experience. The four phases of this didactic engineering we adopted in this investigation (Godino et al., 2014) are as follows:

Preliminary study of the epistemic-ecological, cognitive-affective and instructional (interactional-mediational) dimensions.

Design of the didactic trajectory; selection of problems, sequences and a priori analyses of them, with indication of the expected teacher's behavior and the instructor's planning of controlled interventions.

Implementation of the didactic trajectory; observation of the interactions between the teachers and the resources, and evaluation of the learning achieved.

Retrospective analysis or evaluation, which results from a contrast between what was anticipated in the design and what was observed during implementation. There was also consideration of the rules that condition the teaching process and on the didactic suitability of this process.

The design of this program was established by selecting and adapting activities proposed by the teaching program called Teaching primary school children about probability (Nunes, Bryant, Evans, Gottardis e Terlektsi, 2012), in which we emphasize the approach of beginning with activities on randomness before activities on quantification of probabilities. Furthermore, we selected and adapted general literature activities, considering activities widely used by other researchers in the field of probabilistic thinking. This design phase aimed to develop categories of teacher's didactic-mathematic knowledge (Common Content Knowledge – CCK; Advanced Content Knowledge – ACK and Specialized Content Knowledge – SCK) on probability. The meetings were divided by study units.

4 Example of activity to develop common and advanced knowledge of probability content

4.1 A priori analysis of Activity 9: A bag of candies

Type of problem and practices

In probability activities it is necessary to determine all the possibilities of results of an event in the context of the indicated activities. The combination of all possible results is generally defined as the Sample Space (SS) and plays an essential role for comprehension of the chances and probability of the results of a random event.

The situation-problem we now discuss is part of the set of activities of the Nunes et al. (2012) teaching program; it is based on an example of the classic problem: "a bag contains one white card (W) and two red cards (Ra, Rb), and you can remove two cards at random without replacing them. You may remove two red cards or one red and one white card. Are these two results equally probable or is one more probable than the other?" In the sample space, there are twice as many white-red combinations as red-red combinations because there are four ways to produce the mixed combination (W_Ra, W_Rb, Ra_W, Rb_W) and two ways to produce only the red-red combination (Ra_Rb, Rb_Ra) (Lecoutre, 1992).

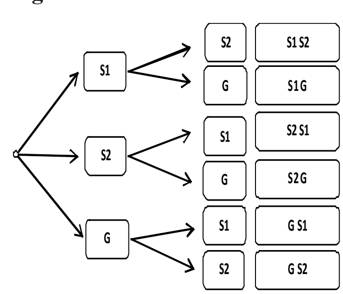
Figure 2 - Item 9 - Situation-problem "A bag of candies"

Samantha can take two candies from a bag, ***without looking****, and there are three candies in the bag. There are two strawberry candies and one gooseberry one. Her favorite flavor is strawberry. She can take two strawberry candies or she can take one strawberry one and one gooseberry one. You can, first of all, guess whether she has a better chance of getting two strawberry candies or a mixture, or decide whether the chance of getting two strawberry candies or a mixture is the same. Think about it and write down your guesses: Why do you think she has a better chance of getting two strawberry candies? Why do you think she has a better chance of getting a mixture? Why do you think the chance of getting two strawberry candies or a mixture is the same?*

Source: Authors.

This situation-problem presents questions that involve the intuitive aspect: Ask them to guess; decide which they think has a greater chance; write down their guesses, etc. Do not suggest the direct use of a formula to calculate the probability, in fact, talk about chance instead of probability. We see that the activity makes them think about all the possible choice possibilities and write down their reasoning to justify the response given. Below (Figure 3) we present an intuitive solution to this situation-problem:

Figure 3 - First solution – intuitive.



To respond to the questions an activity generates, you can observe the result of the pairs for the different candy flavor combinations. Of the six possible combinations, four involve a different mixture of candy flavors in the same way presented in the Lecoutre (1992) version. This demonstrates that the chance of getting a mixture is better, contrary to intuition that can lead to an error due to thinking that having more strawberry candies means that there is a better chance of getting two strawberry candies.

We base our consideration that this solution is intuitive on Fischbein (1993) who states that the intuitive component (intuitive comprehension, intuitive cognition, intuitive solution) relates to an understanding the individual feels is self-evident, leading to acceptance of knowledge or an idea without questioning the need for justification that legitimizes this idea.

The student can determine which combination is more probable without necessarily having to use the probability calculation and check it against his/her response. We will use the Laplace rule to confirm that a mixture is more probable. We call the sample space "SS" and the probability "P."

$$SS = \{S1S2; S2S1; S1G; S2G; GS1; GS2\}$$

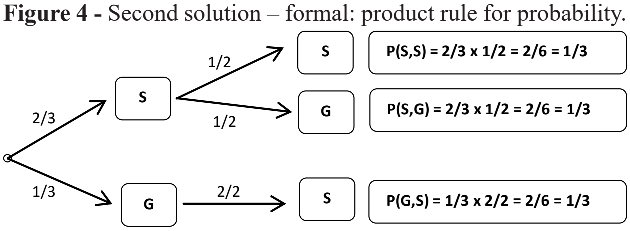
$$\text{(same flavor)} = P(S1S2) + P(S2S1) = 1/6 + 1/6 = 2/6$$

$$P\ \text{(mixture)} = P(S1G) + P(S2G) + P(GS1) + P(GS2)$$

$$= 1/6 + 1/6 + 1/6 + 1/6 = 4/6$$

A second solution described below involves the rule of the product of probabilities and an incomplete tree diagram of probabilities.

Figure 4 - Second solution – formal: product rule for probability.



$$SS = \{S,S; S,G; G,S\}$$

$$P(\text{same flavor}) = P(SS) = 1/3$$

$$P(\text{mixture}) = P(S,G \cup G,S) = 1/3 + 1/3 = 2/3$$

In this second solution, we see that we have a formal component in play. For Fischbein (1993), the formal component relates to knowledge linked to definitions, axioms, theorems and evidences, which should be learned, organized and applied by the student. For him, this component is indispensable in an educational process, since the understanding of what is strict and coherent in mathematics is not acquired spontaneously by the student.

Through this activity we expect teachers to think about the different solutions and the specific probabilities of each solution. Since the choice of taking two candies from a bag of three candies is random in practice, on the first attempt, we still have 1 in 3 chance of taking out any candy; on the second try, there will be:

• 1 in 2 chance, if the first try resulted in a strawberry candy; and there will be

• 2 in 2 chance, if the first try resulted in a gooseberry candy – a certain event.

Therefore, we say that the second attempt is conditioned by what happened on the first, in other words, the second event is dependent on the first.

The teachers involved, undoubtedly covered content on probability during their mathematics teaching undergraduate studies and we assume they studied sample space mapping of an event and the use of combinations to find particular sample spaces, as well as quantification of probability and specific cases, such as conditional and Bayesian probability. This situation requires understanding of the character of certain, impossible, probable, more or less probable, and dependent and independent events, and different ways of mapping sample space, such as the use of diagrams and tables.

Linguistic elements. The linguistic elements that this situation evokes imply the intended institutional meaning, such as the expressions: What is more likely to occur? Which has the best chance? What is the sample space of this event? How can the possibilities be represented? etc. It can be supposed that the teachers involved are familiar with these linguistic expressions of the study of probability (sample space, event, chance, probability).

Conceptual elements. We highlight essential concepts such as "more or less probable events," "possibilities," "chance" and "probability" as concepts that, when studied during the teacher's initial education, assume a characteristic of definition, with an approach only from the procedural point of view. We emphasize that teachers may be familiar with the concept of "sample space" and with "calculation of probabilities." We further note that the ideas of "chance" and "conditionality" of an event can be objects emerging from the practice that one wants to conduct.

Properties. For the development of the activity it is necessary to keep in mind properties (some in an implicit manner), such as:

In some events, having elements in larger quantities does not always imply a higher chance.

Independent events are those where the occurrence or non-occurrence of one does not affect the probability that the other will occur.

Probability of dependent events assumes a restriction of the sample space.

Procedures and arguments. A common procedure is to prepare double entry tables or a tree diagram to map sample space. The tree diagram with probabilities in fractions or percentages is easy for teachers.

It is expected that teachers, in addition to preparing arguments involving the idea of restricted sample space, will build arguments that take into account the fact that having two strawberry candies does not mean there will be a higher probability of actually getting two strawberry candies. It is necessary to deduce that there is a higher probability of getting a mixture and this involves counter-intuitive reasoning.

4.2 Variations of the "bag of candies" situation-problem – development of Advanced Content Knowledge

We expanded this item to the study of conditional probability. Since the Brazilian curricular guidelines (Brasil, 1998) do not include the teaching of conditional probability in the final years of primary education, this knowledge is understood as Advanced Content Knowledge. We believe, however, that in a process of curricular innovation, this topic can also be included in the final years of primary education.

We return briefly to the referenced item and pose two questions related to conditional probability:

There are three candies in a bag. There are two strawberry candies and one gooseberry one. We take two candies out of the bag without looking, one after the other, and do not replace them.

a) What is the probability of removing one gooseberry candy the second time, after having taken out one strawberry candy the first time?

b) What is the probability of removing one strawberry candy the second time, after having taken out one gooseberry candy the first time?

We emphasize that in both cases, we have a context of sample space without replacement, since the composition of the bag once a strawberry candy is removed must be taken into account. The conditional probability implies a restriction of the sample space. Calculation of the conditional probabilities follows:

$$P(G/S) = (P(G,S)) / (P(S)) = (1/3) / (2/3) = 1/2$$

$$P(S/G) = (P(S,G)) / (P(G)) = (1/3) / (1/3) = 1$$

With these probabilities we can infer that it is a certain event (100%) that there will be a mixture, if a gooseberry candy (G) is removed in the first attempt, and that there is a 50% chance of a mixture if a strawberry candy is removed in the first attempt.

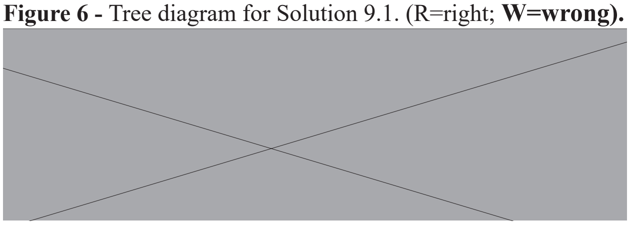
Two more items were then discussed with the teachers, presented as variations of the Bag of Candies situation-problem, in order to mobilize their advanced content knowledge. The following is the item 9.1:

Figure 5 - Description of item 9.1.

A teacher asks his students to answer to true (T) or false (F) questions. *One of the students, Pedro, answers the questions randomly. It is more probable that: i) Pedro answers the two questions correctly (Right); ii) Pedro answers the two questions incorrectly (Wrong); ii) Pedro answers only one of the questions correctly; iv) alternatives i), ii) and iii) are equiprobable.*

Source: Authors.

Figure 6 - Tree diagram for Solution 9.1. (R=right; **W=wrong).**



SS = {RR; RW; WR; WW}

$$P(\text{only one correct answer})=P(RW \cup WR)=1/4+1/4=1/2$$

In this item, the probability of answering only one question correctly (analogue to probability of a mixture) is greater than that of answering the two questions correctly or incorrectly; however, the events are independent.

Thinking about the second removal/try, these activities can be solved in terms of conditional probability (Item 9) in contrast to Item 9.1 (probability involving independent events).

4.3 Description of some potential conflicts

As potential conflicts we highlight the fact that students can confuse sample space with and without replacement when considering that the random removal in Activity 9 would not modify the sample space, not to mention conflicts in identifying the independence or non-independence of the probability experiments. Even teachers have difficulty understanding these concepts.

In a study of Mexican mathematics teachers conducted by Sánchez (2000) on independent events, the teachers presented confused ideas in activities on event independence. They confused independent events with mutually excluding events. Cordani and Wechsler (2006) note that the concept of independent events has caused much theoretical confusion for both students and teachers; in their studies it was common to confuse the word independence with the word excluding, making it difficult to understand these concepts.

Mohr (2008) conducted a study with 122 future mathematics and science teachers enrolled in one or more courses of a mathematics program at a large public university. In his research, when dealing with an item that required calculation of a probability of dependent events, more than half the teachers got it wrong. Three types of the most common errors were observed. First, they did not round off the fraction in the final answer, although it could be argued that the result did not need to be rounded off since it was a question of probability. The other two types were the addition of the two probabilities and treating the item as being one of replacement, even though the problem directly stated "without replacing the first cookie." Mohr (2008, p.36) states: "errors such as these reveal poor understanding of the concepts of independent and dependent events".

Continuing on to discuss the conflicts with conditional probability, we mention the case of the time-axis or transposed conditional fallacies noted by Falk (1986).Of the items discussed above, we have an example of direct conditional probability: What is the probability of removing one gooseberry candy the second time, after having taken out one strawberry candy the first time? And as an example of transposed conditional probability, we have the question: What is the probability of removing one gooseberry candy on the first try, having taken out one strawberry candy the second time?

The transposed conditional fallacy occurs when it is incorrectly assumed that the knowledge of a subsequent event does not affect the probability of what occurred before. If the reader is interested in learning more about this theme, we recommend the studies of Tversky and Kahneman (1982); Gras and Totohasina (1995) and Díaz and De La Fuente (2007). In Activity 9 discussed above, students or teachers can make the mistake of believing that a later event, as in the case of removing a gooseberry candy, if a strawberry candy was removed before, would not affect the final probability. Falk (1986) discovered that, although the students had no difficulty solving direct conditional probability problems (the condition being the occurrence prior to an event whose probability is sought), most of the time they are incapable of calculating whether the condition is posterior to the said event (transposed conditional).

Another more recent study deals with a research study with 196 future primary and high school, and undergraduate teachers on the time-axis fallacy (transposed conditional), conducted by Contreras, Batanero, Díaz and Arteaga (2013). The responses of these teachers to three items related to the time-axis fallacy and Bayes' Theorem were analyzed. The results showed that the time-axis fallacy affected a significant proportion of the participants. The teachers easily solved the item on direct conditional probability, but they had difficulty with the item that asked them to make an inverse inference. These results reinforce those noted by Falk (1986).

5 Example of Activity to Develop Specialized Content Knowledge of Probability

Specialized Content Knowledge is specific to the teacher; for example, to propose mathematics activities, to know how to correct students' errors or even to choose good didactic tools such as a didactic book. To mobilize specialized content knowledge of probability, different strategies were adopted in this formative design, such as responding to items of justification and argumentation and reflecting on activities that present responses constructed by students; a decision similar to that taken in the studies of Gómez (2014) and Ives (2009).

We include here a first example using the strategy of asking justifications. After completing the series of activities on randomness, we asked the teachers questions that required them to provide justifications and arguments. We believe that the approach of justifying and arguing is also part of the epistemic facet of teachers' knowledge. However, the aforementioned questions involve other facets of specialized knowledge as it is indicated in Figure 7.

Figure 7 - Description of activity to develop Specialized Content Knowledge.

1. *Thinking about the series of activities completed, what mathematical content was most evident in this series of activities?* ***(epistemic facet)***

2. *If this series were applied to your students: a) Which activity would elicit most interest? Why? b) Which activity would elicit the least amount of interest? Why?* ***(affective facet) c) What would the difficulties or conflicts be for students in each of the activities?(epistemic and cognitive facet)***

3. *Can this series of activities we engaged in during these two meetings contribute to student understanding of randomness in the final years of primary education?* ***(cognitive facet) Would you make some adaptation? In what way?(meditational facet)***

4. *What different didactic phases can you identify in the entire series completed, considering the interaction between the dynamics of the activity and the students, as well as the interaction between students and teachers when using this activity in the classroom?* ***(interactional facet)***

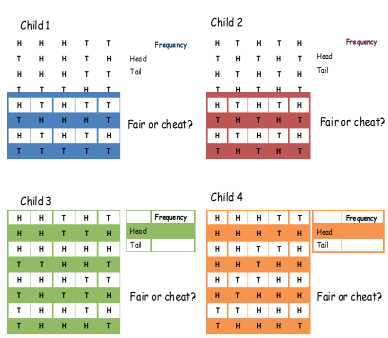
Source: Authors.

Note that the items mentioned above address particular facets of Specialized Content Knowledge. We provide another example by means of the sixth activity included in the formative design – The Case of Coins – of which the first part (6a) is also found in the teaching program of Nunes et al. (2012), and the second part (6b) was designed by us. This is a type of activity in which the teacher has to think about responses constructed by students. Initially, we presented the teachers with the first part of the activity. The teachers can detect predictable patterns in at least one, and, if they observe carefully, in two. Student 2 constructs the H, T, H, T series and so on. Student 4 constructs the 3H, 2T, 3H, 1T series and, thereafter, this series is repeated.

The idea is to discuss predictable patterns during a long series of entries. Patterns can emerge if we look at only a smaller series of entries, but this can occur by chance; it is important to know the difference between a small number of entries and a larger number of entries. This takes into account the law of large numbers systematized by Bernoulli around 1689 as one of the main theorems of probability theory, mentioning that the relative frequency of an event comes closer to the probability of this event when n = "number of repetitions of the experiment," approaches infinity.

Figure 8 - Activity "The case of coins"

First part – 6a) Some children were invited to flip a coin 40 times and record the results. Some of the children did not actually flip the coins and made up their results. Can you tell which children cheated? **(Common Content Knowledge)**



Responses provided by 4 students.

Second part: 6b) How would you explain this difference to 7th grade students, for example? **(Specialized Content Knowledge in the epistemic facet)**

Source: Authors.

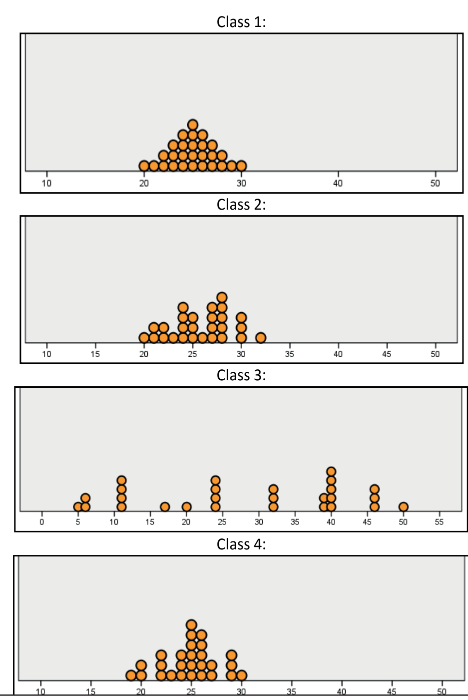
One of the points that should be present in the teachers' arguments is that, as the number of flips increases, the imbalance between heads and tails tends to disappear, which also involves expected probability based on the frequency of results. This first part consists of an activity that can also be presented to students, as proposed in the program of Nunes et al. (2012). So, it is also an example of Common Content Knowledge.

However, in the second part – 6b, the teachers are asked to think about how to explain the difference between a small and large number of flips to students in a specific grade, for example, 7th grade. In this way the teacher appreciates the need to explain the law of large numbers and the meaning of frequency probability, in other words, Specialized Content Knowledge in the epistemic facet. The teacher should know the derivation of the concept of probability and the historical difficulties in constructing this concept, in other words, understand the concept from the epistemological point of view. Errors noted in the history of the concept of probability can be repeated in the classroom by students.

In the next to last meeting, we returned with an activity that is also an example for developing Specialized Content Knowledge. It is designed to analyze arguments and justifications that the teachers offered on the series of random events after observing responses constructed by students. However, this time the activity involved graphs instead of tables.

Figure 9 - Activity "What group cheated?"

20th Activity – Which group may have cheated? Each student in a class should conduct an experiment of flipping a coin 50 times and counting the number of heads. Four different classes produced graphs for the results of this experiment. There is a rumor that students of some classes made a graph without conducting the experiment. Analyze the graph of each class and indicate which class may not have actually conducted the experiment. ***Justify.***



Source: Garfield (2006).

Both activities (6th and 20th) are similar to others used in research on perceptions of randomness by students and teachers (Green, 1983; Batanero, Gómez, Contreras and Gea, 2014).

We highlight another example of an activity to develop Specialized Content Knowledge (cognitive facet) adapted to the studies of Garfield (2006) and Ives (2009). In our list of activities; this is number 19, namely:

Figure 10 - Activity "What car should I buy?"

19) You are trying to decide between two types of cars. So you decide to consult an article in Rodas magazine, which compared the repair rates of several types of cars. Records of repairs made to 400 cars of each type showed a bit fewer mechanical problems for Hondas compared to Toyotas.

You have two friends who have Toyotas and one friend who owns a Honda. Both the Toyota owners mentioned that they have had some mechanical problems, but nothing serious.

The Honda owner, however, gave the following explanation when asked about his car: "First, the fuel injector broke and it cost 750 reals to fix it. Then, I began to have problems with the back of the car that had to be replaced. I finally decided to sell it after these repairs. I'll never buy another Honda."

Given what we now know, which car would you buy? Justify your response.

Teacher: imagine that you are teaching a class and presented this problem to the students. Here are the responses of three different groups of students. How would you proceed with the discussion in class?

a) We recommend that you buy a Toyota, mainly because of all the problems your friend had with the Honda. Since you did not hear of any problems with Toyotas, you should go with that brand.

b) We recommend that you buy a Honda in spite of your friend's bad experience. This is only one case, while the information reported in the consumer articles (magazine article) are based on many cases. According to this data, Hondas are a little more likely to need repairs.

c) We would like to say that it does not matter which car you buy. Even if one of the models may be more likely to have problems than the other, even so, you can simply by chance choose a type of car that will need lots of repairs. You might as well flip a coin to decide.

Source: Ives (2009).

We can see that the response of the letter "a)" group focuses on the information provided by the three friends, in other words, it justifies its response based on the personal experiences of a small group of people. On the other hand, group "b)" justifies the statistical method of the research involving 400 people. Group "c)" disregards both the personal information and that presented by the statistical research, and bets on the random choice mentioned that suggests the decision be made by flipping a coin. Thus, it attributes the idea that both types of car have an equal probability of having defects. In the studies of Ives (2009), this activity was developed to present a teaching situation that involves a real world context. The teachers may have difficulty positioning themselves and arguing, for example, that there is no right or wrong answer. Ives (2009) notes that in his research, the teachers did not feel comfortable and confident about their arguments.

However, we include this activity to develop specialized knowledge in the cognitive facet because the teacher needs to understand student reasoning on the theme of probability to achieve the intended institutional meaning. In this effort to understand what is behind each response, the teacher also reveals his/her knowledge of the theme. If the teacher defends alternative "c)," for example, this can indicate a strong belief that all events are equally likely to occur – one of the errors of equiprobability noted by Lecoutre (1992).

Of all the facets developed within this theoretical framework, we did not illustrate the ecological facet when addressing specialized content knowledge by means of the activities presented in this text. With regard to this facet and the activities discussed above, we can ask ourselves and discuss whether they are adequate, in light of the Brazilian national curriculum and/or that of the state of São Paulo. Or, we can go further and discuss what factors of a social or material nature, or of any other type, condition the use of the referenced activities.

5 Conclusion

In the prior sections we have included and discussed some activities of the formative design that we use with mathematics teachers in Brazil. The examples of activities we discussed here are designed to develop teachers' common and specialized knowledge of probability for better application in their mathematics classes.

The Education Observatory program, through which this education process was implemented, emphasizes the importance of potentially improving mathematics teaching and learning, of probability in our case, in mathematics classrooms.

However, it was not possible to cover all DMK dimensions in this text. One of our limitations arises from the fact that at this point in time we are in the retrospective analysis phase, in particular, with regard to effective implementation of the formative process designed and systematic consideration of the system of restrictions and standards it was subject to. In this phase, the idea of didactic suitability is especially useful, since it can guide recognition of significant improvements of the process developed.

Unfortunately, not all teachers receive good training during their initial education for probability teaching. The series of activities we propose here and it's a priori analysis is based on a contribution that provides adequate support and training for mathematics teachers in the final years of primary education on the specific theme of probability reasoning and its didactics. We believe that teachers should engage in activities that allow them to achieve a higher degree of development of their Didactic-Mathematical Knowledge of probability.

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