CONCEPTUAL AND THEORETICAL ISSUES IN PROPORTIONAL REASONING

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ABSTRACT

This survey of literature on proportional reasoning (PR) focuses on PR studies conducted after a comprehensive review by Tourniaire and Pulos (1985). These connected studies help identify what it means for middle school students to be deemed proficient in PR skills. This paper first describes different existing interpretations of PR and examines a variety of PR problems. It then explores students' development of PR thinking, from their initial acquisition of PR concepts to ways of solving PR problems, as well as their misconceptions about applying PR. The paper concludes by discussing ways to use this literature review to develop a PR assessment from a cognitive diagnostic model framework.

Keywords: ratio, proportion, proportional reasoning, multiplicative reasoning, solution strategies, misconceptions.

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INTRODUCTION

Proportional reasoning (PR) is an important form of mathematical thinking and the foundation on which high levels of mathematical knowledge in other domains are built. Lesh, Post, and Behr (1988) assert that PR is the capstone of elementary school mathematics and the cornerstone of high school mathematics. In the *Curriculum and Evaluation Standards for School Mathematics*, the following was stated:

> *The ability to reason proportionally develops in students throughout grades 5-8. It is of such great importance that it merits whatever time and effort must be expended to assure its careful development. Students need to see many problem situations that can be modeled and then solved through proportional reasoning. (NCTM, 1989, p. 82)*

Despite its importance, Lamon (2007) notes that PR research was at a temporary standstill due to the loss and unfinished projects of many great mathematics educators such as Merlyn J. Behr in 1995, Leen Streefland (Netherlands) in 1998, and Robbie Case (Canada) in 2000. Lamon also cited Davis and colleagues (1993), who firmly believe that a common ground in the field of rational number is lacking and "no real progress is being made" (p. 63). According to Crick (1988, as cited in Davis et al., 1993), who referred to literature development in biological science at that time, this standstill stems from researchers of one subgroup failing to cite the results of others of different subgroups, creating "a number of somewhat separate schools" (p. 63). As we will argue through descriptions of interpretations, task constructions, and the development of mathematical thinking in PR, such a bold critique by Davis and colleagues has become ever more applicable to research on PR.

As a first step to address this predicament, it is necessary to understand PR and its underlying concepts better through a comprehensive coverage and treatment of PR. Reviews in the two research handbooks published by NCTM (Behr et al., 1992; Lamon, 2007) did not focus exclusively on PR. In fact, such a comprehensive review, as done by Tourniaire and Pulos (1985), has not been completed for over two decades; thus, our discussion of PR is based primarily in studies conducted since 1985.

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We surveyed PR literature with the expectation that developments therein can provide a theoretical underpinning for new innovative assessments that can inform classroom instruction and student learning. We intend to use the findings from this review to develop a PR assessment from a cognitive diagnosis model framework. In contrast to traditional psychometric models, which are useful in locating students along a single proficiency continuum, cognitive diagnosis models are designed to provide information on students' mastery or nonmastery of finer-grained skills, cognitive processes or problem-solving steps, generically referred to as attributes (de la Torre, 2009; de la Torre & Lee, 2010). Based on past research, we attempt to shed light on PR attributes, considering that these attributes may come from analyzing how PR tasks are constructed as well as how student thinking in PR is developed. Understanding these attributes is essential to explain student success and failure when acquiring PR and to serve as a foundation for developing effective tools to assess students' future understanding of PR.

INTERPRETATIONS OF PROPORTIONAL REASONING

The interpretation of PR broadly relies on definitions of proportion, which in turn is dependent on how ratio and rate are defined. Among many disagreements on the distinction between ratio and rate (Kaput et al., 1986; Lesh et al., 1988; Ohlsson, 1988; Schwartz, 1988; Thompson, 1994) was Vergnaud's (1983) most conventional interpretations: *ratio* is a comparison of two quantities with similar measures, whereas *rate* is a comparison of two quantities with unlike measures, e.g., *a ratio of 3 girls to 2 boys*, and *a rate of 4 miles in 5 hours*. Even with a distinction between ratio and rate, there is no agreement on labeling the same idea with a common term. Schwartz (1988) advocated replacing the term ratio or rate with *intensive quantity* to indicate a relational quantity composed of two *extensive quantities*, e.g., an intensive quantity of *$6 per pound of coffee* relating two extensive quantities: *$6 cost* and *1 pound weight of the coffee*. Building on the concepts of rate and ratio, a *proportion* is defined as a mathematical equivalence of two ratios or rates, e.g., $\frac{3\,\text{girls}}{2\,\text{boys}} = \frac{6\,\text{girls}}{4\,\text{boys}}$.

Like other compositions of mathematical symbols, a proportion can be thought of

formalizing the expression of an abstract relationship into a standardized mathematical statement.

This mathematical formalization of a proportional relationship is not entirely synonymous with PR. Indeed, PR requires understanding a *second-order relationship* (Piaget & Inhelder, 1975, as cited in Lesh et al., 1988): it is not enough to understand one relationship about two quantities since PR involves a relationship (i.e., proportion) between two relationships (i.e., ratios or rates). Karplus et al. (1983a; Stanley et al., 2003b) offered an alternative interpretation of Piaget's definition of second-order relationship by restricting proportionality to a linear functional relationship with *direct variation* (i.e., $y = kx$, where *k* is a *constant ratio* that relates quantities *x* and *y*), contrasting it with that with direct variation and *constant difference* (i.e., $y = kx + l$, where *l* is the *constant difference* applied to quantity *y*) and that with *inverse variation* (i.e., $y = \dfrac{k}{x}$, where *k* is a *constant product* that relates quantities *x* and *y*).

Lesh et al. (1988) defined PR as "a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information" (p. 93). In their view, PR needs to be differentiated from *pre-PR*. The latter is characterized by the inability to recognize the structural similarity of two equivalent relationships of a given proportion, as well as the erroneous use of additive reasoning and the blind application of cross-multiplication. These authors agreed with Tourniaire and Pulos (1985) and Karplus et al. (1983a, 1983b) that PR evolves through a gradual increase in local competence: young children cultivate mastery of PR skills in small areas of inquiry and problem settings, and progressively extend their competence to larger contexts of problem solving. This view contrasts with the Piagetian view of PR as a global ability related to general cognitive structure.

Lamon (2007) suggested that PR means:

> *supplying reasons in support of claims made about the structural relationships among four quantities, (say a, b, c, d) in a context simultaneously involving covariance of quantities and invariance of ratios or products; this would consist of the ability to discern a multiplicative relationship between two quantities as well as the ability*

to extend the same relationship to other pairs of quantities (p. 637-638).

Yet, she favored the term *proportionality* as being far more advanced than an understanding of rational number or PR and suggests that it develops as one studies higher levels of mathematics and science. To Lamon, proportionality is "a mathematical construct referring to the condition or the underlying structure of a situation in which a special invariant (constant) relationship exists between two covarying quantities (quantities that are linked and changing together)" (p. 638).

Drawing from all these interpretations, one may deduce that PR incorporates three essential abilities to comprehend: 1) a first order rational relationship within two quantities or measures via a constant ratio, 2) a second order proportional relationship between two ratios or rates via a constant multiple, and 3) a variety of applications and representations of the structure and concept of proportional relationships.

TASK CONSTRUCTIONS OF PROPORTIONAL REASONING PROBLEMS

PR problems usually take the form of: 1) a missing value problem (MVP), 2) a comparison problem, or 3) a qualitative problem (see Appendix for exemplars of problems). In the first, students are given one ratio in a proportion and one quantity of the second ratio and asked to determine a missing quantity in the latter (e.g., *Problem 1*). In a comparison problem, students are given two ratios and asked whether one is less than, greater than, or equal to the other (e.g., *Problem 2*). A qualitative problem (or a relatively problem; Freudenthal, 1983) requires students to consider the effect of an increase or a decrease in one part of a proportion (e.g., *Problem 3*). Difficulty level associated with each type of PR problems depends on its structural and contextual components.

Structural components of missing value problems

One structural component of a MVP is the *location* (*order*) of the missing value, which can be *a*, *b*, *c*, or *d* in the proportion $\frac{a}{b} = \frac{c}{d}$. Harel and Behr (1989)

believed that students can find missing values on the right side of a proportion, particularly *d*, more easily than those on the left side. Another structural component of a MVP is the *units of measure*, which can be considered in terms of measure space involved, dimension of units, and partitionability of units (Harel & Behr, 1989). MVPs can involve one measure space only (e.g., *Problem 4*), two measure spaces presented together (e.g., *Problem 5*), or two measure spaces presented as linked, (e.g., *Problem 1*). Measure spaces can also be presented in different dimensions, e.g., *Problem 6* involves a consistent measure space, but different dimensions within that measure space, whereas *Problem 7* involves two measure spaces, but different dimensions within those measure spaces. Students considered problems with varying dimensions more difficult than those with consistent dimensions (Harel & Behr, 1989). In regards to partitionability of units, i.e., discrete (e.g., number of apples) vs. continuous measure spaces (e.g., cost of apples), Harel and Behr (1989) hypothesized that unpartitionable units would be more difficult for students to conceptualize. Moreover, measure spaces of MVPs can be coordinated (e.g., *Problem 7*) or uncoordinated (e.g., *Problem 1*). Conner et al. (1988) found that the order of the MVP and the coordination of measure spaces may interact when contributing to problem difficulty: the difficulty for the order of the missing value varied according to whether the measure spaces were coordinated.

Numerical relationships in a problem also affect problem difficulty. Kaput and West (1994) observed that reduced ratios and familiar multiples aided in solving PR problems, unlike non-integer multiples which had an opposite effect. They found that when posed with integers very close in value, as opposed to those with a large difference, students more likely used incorrect additive strategies. Niaz (1988) showed that some ratio pairs suggest additive relationships more than others, e.g., when comparing *4* to *6* and *6* to *x*, students more likely provided an incorrect additive response, *8*, than when comparing *3* to *5* and *4* to *x*.

Structural components of comparison problems

While the order of MVP is not applicable to comparison problems, presentation and coordination of measure spaces and numerical factors are relevant. Yet, few researchers investigated the unique structural components of comparison problems.

The literature we found on comparison problems focuses mainly on studying numerical characteristics of the problems. For example, Karplus et al. (1983a) found that presence of integer ratios combined with presence of unit ratios can help children to solve proportional comparison problems, especially when integer ratios are given both within and between ratios (e.g., *Problem 8*).

Alatorre (2002) analyzed structure of comparison problems in the context of probability. Participants were presented with two probabilistic situations (i.e., ratios) and asked to choose which had the higher probability of getting the desired result. Alatorre claimed that difficulty of probability comparison situations is affected by such characteristics of each ratio as: number of favorable and unfavorable elements, total number of elements, and differences between favorable and unfavorable elements, and magnitude of probabilities. A detailed framework was developed and tested on college students. Much like those who studied the difficulty of MVPs, Alatorre found that the components in her framework interacted in a complex way to affect problem difficulty.

Contextual components of PR problems

In describing the context of each ratio within a PR problem, Lamon (1993) discussed four semantic types: 1) *well-chunked measures*: a comparison of two extensive measures conceptualized as an intensive concept (e.g., *miles per hour* as *speed*), 2) *part-part-whole*: each ratio in a proportion consisting of two subsets that were part of a larger set (e.g., *a ratio of girls to boys to all children*), 3) *associated sets*: the relationship between the two compared elements defined in the problem context, and 4) *stretchers and shrinkers*: scaling an object up or down. Lamon suggested that associated sets were the easiest, followed by part-part-whole, well-chunked measures, and stretchers and shrinkers. Cramer and Post (1993) also found stretcher and shrinker problems (*scaling problems*) to be the most difficult one. Some researchers note that *mixture problems*, a particular type of part-part-whole problem, are especially difficult for students (Alatorre & Figueras, 2005; Tourniaire, 1986).

Students' *context familiarity* with PR problems also affects their performance (Tourniaire, 1986). Saunders and Jesunathadus (1988) found that students

performed significantly better on familiar context problems (e.g., the cost of items at a store) than unfamiliar ones (e.g. high school chemistry), despite the fact that both types of problems contained similar proportional relationships and even used similar numbers. Context familiarity, however, can also bring unforeseen difficulties. Karplus et al. (1980) found that when solving animal context problems, middle school students tended to write responses that focused on the animals in the problem rather than on the animals' numerical characteristics. Peled and Bassan-Cincinatus (2005) found that *Problem 9* elicited debate: while some participants believed they could solve the problem using PR (i.e., Anne gets $24 and John gets $16), others believed that the winnings should be split fairly between Anne and John. Although acknowledging that this was not a mathematically sound answer, the participants argued that splitting the winnings equally was more morally acceptable. Hence, context familiarity may help students solve PR problems, but in certain cases, it may also distract them from the mathematics.

Researchers have also found that students with access to *manipulative materials* perform better on PR problems. Fujimura (2001) found that letting students use manipulatives to reason through comparison problems may be more effective than simply teaching them strategies (e.g., comparing unit quantities). Manipulatives can also be useful when students are checking their answers to PR problems (Kwon et al., 2000).

STUDENT THINKING: TOWARDS IDENTIFYING ATTRIBUTES OF PROPORTIONAL REASONING

Many studies have been devoted to explain how young students think proportionally. Kieren (1988) showed that children as young as age 5 understood basic understanding of PR problems involving continuous quantities, significantly more than those involving discrete quantities. Boyer et al. (2008) suggested that the presence of even one continuous quantity can help children reason proportionally, whereas that of two discrete quantities can hinder PR. Jeong et al. (2007) found that when students of ages 6, 8, and 10 are presented with discrete quantities, they may

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feel inclined to count discrete sections rather compare overall ratios. Schwartz and Moore (1998) found that sixth graders presented with the numerical juice mixture task felt the need to make unnecessary calculations, thus performing significantly worse than those presented with the non-numerical task. Ahl et al. (1992) found that performance on non-numerical problems varied little across age groups, whereas older participants tended to perform better on numerical problems.

Students' proportional reasoning strategies

When young students approach a PR problem, they may use a *building-up strategy* to arrive at a solution (Kaput & West, 1994; Lamon, 2007), e.g., *7 apples for $5*, *14 apples for $10*, and *21 apples for $15* (*Problem 1*). Similarly, students can use this strategy when given a particular ratio and asked for a smaller value (i.e., *building-down*; *iterating composite units*, Battista & Borrow, 1995). When students abbreviate repeated addition as multiplication, they are using an *abbreviated building-up strategy* (Kaput & West, 1994; or *linking iterative composite units*, Battista & Borrow, 1995), e.g., since there are 3 groups of 7 apples in 21 apples, the cost is $5, $10, and finally $15 (*Problem 1*). A *formal division* may be used in the abbreviated building-up process (Kaput & West, 1994), e.g., 21 apples divided by 7 apples is 3 times, so 3 times $5 is $15 (*Problem 1*). Clearly, such particular building-up strategies are most appropriate for problems involving integer ratios.

Nonetheless, children comfortable with the building-up strategy must find new ways of reasoning for problems in which given ratios cannot be repeatedly added evenly to obtain ratios in question. Kaput and West (1994) described how students adjusted the building-up strategy via an *early adjustment strategy*: since 14 apples for $10 means 7 apples for $5, building-up results in 35 apples for $25 (*Problem 10*). Alternatively, students may use a *late-adjustment strategy* (Kaput & West, 1994): since 14 apples cost $10, building up results in 28 apples for $ 20, adjusting makes 7 apples for $5, and so 35 apples will be $15 (*Problem 10*). Lamon (1994; 2007) described the strategy of *norming* to solve comparison problems with non-integer ratios such as *Problem 2*: considering one ratio in terms of another (i.e., the ratio of $15 to 15 oranges in terms of that of $5 to 7 apples) results in the observation that

either two $5's of 14 oranges leaves $5 for 1 orange, or $5 could buy 6 additional apples, rather than only 1 orange.

While norming and adjusted building-up strategies are considered correct strategies, students using them are not considered to be reasoning proportionally (Kaput & West, 1994; Lamon, 2007). Moreover, norming and adjustment processes for solving problems without integer ratios can be complicated. As students develop multiplicative ideas, they find even more efficient ways to approach PR problems. Yet, moving from informal strategies to multiplicative reasoning is not necessarily easy. Battista and Borrow (1995) observed that their case study participant had difficulty iterating composite units (i.e., building-up) to represent her thinking formally in terms of multiplication and division. They hypothesized that for the student to reason multiplicatively, she would need to: (a) "explicitly conceptualize the repeated action of linking the two composites" (p. 5-6), (b) understand the concepts of multiplication and division well enough to see their roles in the iteration process, and (c) abstract the iteration process to reflect on it before using multiplication and division.

One way to reason multiplicatively is through a *unit-factor approach* (Kaput & West, 1994; Christou & Phillipou, 2001; *unitizing*, Lamon, 2007; *unit-rate strategy*). Kaput and West (1994) argue that this approach is a natural extension of the adjustment strategies. As students develop this approach, they may choose units that are easier to manipulate (Lamon, 2007), e.g., 7 apples is a preferred unit and a common multiple of 14 apples and 35 apples (*Problem 10*). This approach is considered a multiplicative strategy because it requires not only recognizing an invariance of ratio, but also understanding of multiplicity of equivalent ratios to another ratio. However, Singh (2000) demonstrated that students using unit-factor approach do not necessarily understand the multiplicative nature of PR.

Another way to reason multiplicatively is through a *factor-of-change strategy*, especially to solve PR problems with integer ratios (Cramer & Post, 1993), e.g., since there are 3 times as many 7 apples as there are 21 apples, 21 apples cost 3 times as much as 7 apples (*Problem 1*). Lamon (1994) described two specific types of reasoning within the factor-of-change method: 1) *within strategy* (*scalar method*) in comparing two quantities within one measure space and applying a scalar factor to

find a missing quantity in another measure space (e.g., a scalar factor of 3 within 7 apples and 21 apples [*Problem 1*]), and 2) *between strategy* (*functional method*) in evaluating a functional relationship between a quantity in one measure space and its corresponding quantity in another measure space (e.g., a functional relationship of $\dfrac{\$5}{7\,\text{apples}}$ between \$5 and 7 apples [*Problem 1*]). Post et al. (1988) argue that students can use the factor-of-change strategy by considering graphical representations and interpreting the slope of a straight line passing through the origin as the unit rate of the problem (e.g., the slope $\dfrac{5}{7}$ in *Problem 10*).

Misconceptions in proportional reasoning

An *additive strategy* (*constant difference strategy*) is an incorrect strategy that relies on differences between quantities within a ratio, typically as a result of a partial understanding of building-up strategies, e.g., since the difference between 35 apples and 14 apples is 11, 35 apples cost \$11 more than 14 apples do (*Problem 1*). Although once thought to be widely used by students who did not reason proportionally, research revealed that additive strategy is often used as a fall-back strategy to solve PR problems involving non-integer multiple ratios (Lesh, et al., 1988; Karplus & Karplus, 1972; Karplus, et al., 1983a; Tourniaire, 1984).

A *mismatch of numerator and denominator between ratios* in a proportional relationship occurs more frequently in PR problems involving uncoordinated measure spaces (Conner et al., 1988), e.g., reading the numbers in an orderly fashion may result in an incorrect construction of proportion $\dfrac{7}{5}=\dfrac{x}{21}$ (*Problem 1*). A variation of this misconception is observed in comparison or qualitative problems where a lack of PR results in *comparing only the value of numerator between two ratios without attending to the value of denominator*, e.g., since \$15 is greater than \$5, oranges are more expensive than apples (*Problem 2*).

Many researchers (Farrell & Farmer, 1985; Stanley et al., 2003a) observed that, when solving in problems involving inverse proportional relationships, e.g., *Problem 11*, students often have difficulty in *differentiating between situations where*

PR is appropriate and those where it is not. Post et al. (1988) and Lanius and Williams (2003) believe that students may also fall into assuming inappropriate proportionality across a linear relationship with a constant difference, e.g., an entry cost of $2 (*Problem 12*). Another case of blind applications of direct proportionality in a non-linear relationship is a phenomenon called *pseudo-proportionality* (Modestou & Gagatsis, 2007), e.g., *Problem 13*.

Transitioning into mathematical formalization

Because of its clear way to represent PR problems that facilitates an efficient approach, formal algebraic proportions are widely used throughout secondary school and adulthood across subject areas in applied contexts (e.g., Carraher et al., 1988). One might suspect that students' proficiency with rational numbers (in particular, finding equivalent fractions) can affect their ability to solve formal proportions. In investigating the relationship between PR and fraction skills, Heller et al. (1990) suggest that fraction proficiency implies successful performance in PR, but not the reverse.

Still, much of the PR research has stressed the need for students to be proficient in using multiplicative strategies before being introduced to abstract strategies such as cross-multiplication (Kaput & West, 1994; Lamon, 2007). Failure to connect intuitive ideas to formal proportions can cause misunderstandings. Heller et al. (1990) found that seventh graders used the unit-factor approach more often than the factor-of-change strategy or cross-multiplication, whereas eighth graders used cross-multiplication more often than other strategies. However, unlike seventh graders, eighth graders also applied cross-multiplication to non-proportional problems (Cramer & Post, 1993), possibly because cross-multiplication was taught in school between seventh and eighth grades, but students did not automatically connect their intuitive thinking to these abstract strategies. Kaput and West (1994) argue that this disconnect stems from informal strategies allowing students to maintain an understanding of the relationship between referents, whereas formal strategies may produce measure spaces unfamiliar to students, e.g., cross-multiplication may yield 105 apples $ (*Problem 1*).

At this point, one may question the advantage or practicality of helping younger children establish the formalization technique of writing a proportion. Given the multiple correct strategies with which one can approach PR problems, forming ratios with an equal sign between them does not directly constitute the ability to reason proportionally, but still, it is impossible to construct a proportion without prior knowledge of ratios. Thus, one cannot avoid presenting the concept of ratio before that of proportion in the curriculum, although both concepts are more suitable when instructed after initial ideas about PR (Smith, 2002). Young children should receive opportunities to reason about proportional situations prior to instruction in concepts of ratio and proportion. Since the concept of proportion merely formalizes that of PR, it is advisable to institute formal algebraic procedures in solving proportion problems only after qualitative and informal quantitative approaches have been acquired successfully (Christou & Phillipou, 2002; Kaput & West, 1994; Lamon, 1994).

DISCUSSION

The review of these studies has made it clear that many leading researchers who specialize in the field of PR, worked on identical experiments, and concluded several analogous points of view, yet remain unenthusiastic about synchronizing their perspectives. What was true in biological science research according to Crick (1988, as cited in Davis et al., 1993) echoed the literature on fraction learning as maintained by Davis and colleagues (1993) and finally appeared to be a self-fulfilling prophecy in research on PR. This fractured state has contributed to a "temporary standstill" (Lamon, 2007). We have attempted to present the different research areas and disagreements in the field while reconciling them into a coherent picture of PR.

We also anticipate this review will trigger discussions on the specific, measureable attributes of PR. Understandably, acquisition of PR is a process, rather than a product, of teaching and learning, comprised of a set of skills which may be ordered or unordered, replaceable or irreplaceable, and separable or collective of other sets of skills. One could hope for a comprehensive compilation of skills and processes, or sets of skills and processes, as a way to afford generalization of PR acquisition across different studies. Accordingly, such compilation could be geared

towards concrete quantitative, rather than abstract qualitative, measurements of PR skills. This compilation consists of a set of skills and/or processes that can facilitate mathematics educators to pinpoint students' deficiencies in specific skills or processes needed to master PR. This approach is arguably more relevant to improving instructional practices than simply knowing a student's proficiency, or lack thereof, based on a single overall score and a predetermined cut-score (de la Torre, 2009).

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APPENDIX

Proportional Reasoning Problems

Problem 1: Seven apples cost $5. How much do 21 apples cost?

Problem 2: Seven apples cost $5 and 15 oranges cost $15.Which fruit is more expensive?

Problem 3: Al has 7 apples and 8 oranges. If Al ate some of his oranges, would the ratio of apples to oranges become greater than, equal than or less to the original ratio?

Problem 4: Five of the 7 fruits in a basket are apples. What percent of the fruits are apples?

Problem 5: Al buys 7 pens and Eve buys 14 pens. Al pays $5. How much does Eve pay?

Problem 6: When Al runs 5 km, Eve runs 500 m. How many meters does Eve run when Al runs 5.7 km?

Problem 7: Al runs 7 km in 5 minutes. How many kilometers does he run in 1 hour?

Problem 8: One apple costs $5 and 6 oranges cost $30. Which fruit is more expensive?

Problem 9: Two friends, Anne and John, bought a $5 lottery ticket together. Anne paid $3 and John paid $2. Their ticket won $40. How should they share the money?

Problem 10: Fourteen apples cost $10. How much do 35 apples cost?

Problem 11: A picture of 4 by 6 has the same area as a picture of 3 by what?

Problem 12: Apple picking tour costs $2 per entry and $5 for every 7 apples picked. How much does Al pay for 21 apples on an apple picking tour?

Problem 13: There are 12 inches in 1 foot. How many square feet are there in 432 square inches?