INVESTIGATING MATHEMATICAL COGNITION USING DISTINCTIVE FEATURES OF MATHEMATICAL DISCOURSE

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ABSTRACT

Ben-Yehuda, Lavy, Linchevski, and Sfard (2005) propose that there are four distinctive features of mathematical discourse: uses of words, the use of uniquely mathematical visual mediators, special discursive routines, and endorsed narratives. Utilizing modified talk- and think-aloud protocols for research in naturalistic settings this research explored: (1) the potential utility of the proposed four distinctive features of mathematical discourse as an analytic tool in studying mathematical cognition and mathematical cognitive processes, and (2) the interaction (or lack of interaction) between the four distinctive features of mathematical discourse in order to contemplate implications to students' underlying or emergent mathematical cognition. Our findings suggest that there may be a unique interaction between particular features of mathematical discourse, and an absence of this interaction may be an important indicator for additional support for the learner. Implications to teaching, learning, and future research will be discussed.

Key words: discourse; learning; mathematical cognition; mathematical processes; talk-aloud; teaching; think-loud.

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INTRODUCTION

An understanding of students' mathematical cognition and associated mathematical cognitive processes is crucial for shaping instructional practices in such ways that advance student learning. Numerous scholars have used mathematical discourse in various ways as windows or lens into students' mathematical cognition/processes (Gustafson & MacDonald, 2004; Lerman, 2001; Mercer, Wegerif, & Dawes, 1999; Nührenbörger & Steinbring, 2009; Radford, 2004). For instance, Sfard and Kieran (2001) studied mathematical discourse to determine the extent to which interactions between students supported individual student learning and thinking. Their study of dyads showed that some interactions, under certain conditions, were not beneficial for developing mathematical cognition. For the purpose of this research, we adopt an elaborated definition of mathematical discourse to include "all forms of language, including gesture, signs, artefacts, mimicking, and so on" (Lerman, 2001, p. 87), that are used for the purpose of engaging in and with mathematics.

From a theoretical perspective, using mathematical discourse as a window into mathematical cognition is supported by numerous scholars and theorists (Lerman, 2001; Radford, 2004; Sfard & Kieran, 2001). This support stems largely from the cognitive theories of Vygotsky (1962) who described talking aloud as the manifestation of inner thought or cognition which includes both verbal and non-verbal mental representations (i.e., visual mediators). Vygotsky theorizes that as children "solve practical tasks with the help of their speech, and action, which ultimately produces internalization of the visual field" (p. 26). Further, cognition "is not merely expressed in words; it comes into existence through them" (Vygotsky, 1962, p. 125).

Mathematical cognition, as used in this research, is intended to describe mathematical thinking, knowledge, and/or understanding (e.g., algebraic, geometric, numeric, etc) (Campbell, 2005). Whereas, mathematical processes are those cognitive processes that describe how mathematical cognition is either acquired or acted upon (Campbell, 2005). For example, according to the NCTM (2000) there are five different mathematical processes identified within the "Process Standards" : (1) problem solving, (2) reasoning and proof, (3) communication, (4) connections, and (5) representation. Simply stated, mathematical cognition describes the "what" aspect of the underlying thinking and mathematical processes describe the "how." Often the two concepts, mathematical cognition/processes, are intertwined and thus can be observed concurrently.

Recently, Ben-Yehuda, Lavy, Linchevski, and Sfard (2005) proposed that there are four distinctive features of mathematical discourse:

> *(1) uses of words [authors' italics] that count as mathematical; (2) the use of uniquely mathematical visual mediators [authors' italics] in the form of symbolic artefacts that have been created specifically for the purpose of communicating about quantities; (3) special discursive routines [authors' italics] with which the participants implement well-defined types of task; and (4) endorsed narratives[authors' italics], such as definitions, postulates, and theorems, produced throughout the discursive activity. (p. 182)*

In comparison to other taxonomies of mathematical discourse that largely describe types of communication patterns in classrooms (cf. Mercer, 1996; Mercer et al., 1999; Pirie, 1998)[4], we hypothesize that the four distinctive features proposed by Ben-Yehuda, Lavy, Linchevski, and Sfard may potentially provide a useful cognitive framework for analyzing mathematical cognition, mathematical processes, or both, depending on the mathematical context in which they are used (Duval, 2006; Halliday, 1978; Moschkovich, 2003; Pimm, 1987; Sfard, 2000; Winslow, 1998).

Situated against the elaborated definition of mathematical discourse stated earlier (i.e., to include gestures, postulates, routines, words, visual mediators, etc.), each of these distinct features of mathematical discourse may be able to provide evidence of students' mathematical cognition (i.e., mathematical thinking) and/or mathematical processes. For example, the creation of a graph (i.e., a visual mediator), may simultaneously be representative of specific mathematical cognition (e.g., visual representation of numerical relationships). At the same time, a graph, as a visual mediator, may also be reflective of a mathematical cognitive process (i.e., the "how") if the graph is used to generalize, hypothesize, and so forth, about mathematical meaning (See Table 1 for additional examples).

[4] Pirie (1998) describes the following types of communication in a mathematical classroom: *Ordinary* language, m*athematical verbal language, symbolic language, visual representation, unspoken but shared assumptions,* and q*uasi-mathematical language*. Mercer (1996) describes three types of mathematical discourse: *disputational talk, cumulative talk, and exploratory talk*.

Table 1. Examples of mathematical cognition/processes from the Ontario Elementary Mathematics Curriculum (2005) for each of the four distinctive features of mathematical discourse.

Feature of mathematical discourse	**Example of mathematical cognition**	**Example of mathematical processes**
Uses of words	Explain the relationship between a census, a representative sample, sample size, and a population (e.g., "I think that in most cases a larger sample size will be more representative of the entire population.").	Make inferences and convincing arguments that are based on the analysis of charts, tables, and graphs (Sample problem: Use data to make a convincing argument that the environment is becoming increasingly polluted.).
Visual mediators	Represent linear growing patterns (where the terms are whole numbers) using graphs, algebraic expressions, and equations.	Model linear relationships graphically and algebraically, and solve and verify algebraic equations, using a variety of strategies, including inspection, guess and check, and using a "balance" model.
Special discursive routines	Represent, compare, and order equivalent representations of numbers, including those involving positive exponents.	Solve problems involving whole numbers, decimal numbers, fractions, and integers, using a variety of computational strategies.

Endorsed narratives	Develop geometric relationships involving lines, triangles, and polyhedra, and solve problems involving lines and triangles.	Demonstrate an understanding of the geometric properties of quadrilaterals and circles and the applications of geometric properties in the real world.

As such, our goals in this research were as follows: (1) to explore the potential utility of the proposed four distinctive features of mathematical discourse as an analytic tool in studying mathematical cognition and mathematical cognitive processes, (2) to examine the interaction (or lack of interaction) between the four distinctive features of mathematical discourse proposed by Ben-Yehuda and colleagues in order to contemplate what potential interactions might suggest about students' underlying or emergent mathematical cognition/processes, and (3) to reflect upon implications of our findings to teaching, learning, and future research.

To achieve these research goals, six eighth-grade children were invited to document their mathematical cognition and mathematical cognitive processes while completing their homework at home using *mathcam video diaries*. Although the focus of this research is not homework completion, per se, understanding mathematical cognition/processes used during homework may nevertheless be a corollary benefit to this research given that success in completing homework has been found in numerous studies to increase mathematical achievement (Cooper, Robinson, & Patall, 2006; Fife, 2009; Hong, Peng, & Rowell, 2009).

METHODS

Ericsson and Simon's (1993, 1998) talk- and think-aloud protocols were used to examine the relationship between Ben-Yehuda, Lavy, Linchevski, and Sfard (2005) four distinctive features of mathematical discourse. Particularly relevant to the present research are Ericsson's and Simon's (1993, 1998) protocols associated with *Level 3 verbalizations*, which are verbalizations linked to instructions to explain or describe their thinking either retrospectively or concurrently to task completion. In the present research, both applications apply. Students may be verbalizing as they concurrently engage in mathematics homework completion or may retrospectively describe prior mathematical learning or experiences.

Faithful appropriations of Ericsson and Simon’s (1993, 1998) talk- and think-aloud protocols occur exclusively in experimental settings. In later explications of their model for analyzing thinking and talk, Ericsson and Simon proposed that everyday situations can be reproduced in controlled laboratory settings. Indeed, many researchers simulate classroom learning with laboratory “training” (Anderson, Reder, & Simon, 2000, Summer). As Anderson, Reder, and Simon explain, learning as a complex skill is hierarchical in structure with multiple nested components that require both analyses in the laboratory and in real-world settings. The application of Ericsson and Simon’s protocols in a naturalistic setting is a strength of this research.

Participants

Data for this paper were drawn from a year-long study investigating the home-school connection in mathematics learning in an eighth grade classroom. The teacher involved in the research (third author) had been teaching 12 years and had completed a master’s degree. He was approached to participate in this research and he agreed.

The school was located in a diverse urban setting. Duane’s class consisted of 28 students, 14 male, and 14 female. All students were either 13 or 14 years of age. From these 28 students, a purposive sample of six students (three males and three females) were selected and invited to participate in the mathcam video diaries (videotaping their mathematical thinking in their homes during homework completion) portion of the research. English was the first language of all but one of the students.

In consultation with Duane, the six students invited to participate in the mathcam video diaries portion of the research were selected based upon the following considerations: (1) perceived ability of the student by the teacher to engage in thinking aloud, (1) perceived level of responsibility of the student to maintain continued engagement with the student and to care for the home equipment, (3) gender, and (4) ability. The goal in terms of ability and gender was to ensure mixed representation. In addition to the six students that were selected, two alternates were also selected in the event that a student had to withdraw from the research, which was the case with one of the six students initially selected.

In order to focus the reporting of the results, excerpts from the mathcam video diaries were drawn specifically from one of the six students – Kara who was 13 years old at the time. Kara achievement results from the seventh grade indicated that she was a B-level student. She described herself as a hard worker and responsible. She was the youngest of four children and lived with both her parents. English was her first language. We selected mathcam video diaries submitted by Kara because of the explanatory potential of the excerpts. Additionally, her mathcam video diaries were seen as representative of the sample analyzed.

Data sources

Mathcam video diary data from the six students invited to participate in the mathcam video diaries portion of the research were the primary sources of data. These six students, their parents, the classroom teacher, and the research team, met to engage in a training session and to distribute the laptop computers which were provided for the duration of the research to each of the six students. At that time, the six students (and their parents) were trained on (a) how to video record their verbalizations using cameras built-into the laptop computers, (b) how to transfer their recordings using secure email or memory sticks, and (c) how to engage in the task of talking aloud about their mathematical cognition.

We encouraged the six students to document everything they were thinking and doing in order to assist themselves with the understanding and completion of the mathematical homework. We anticipated one to five submissions per week from each of the six students invited to participate in the mathcam video diaries portion of the research.

For this research we analyzed 34 mathcam video diaries (mean length 4.79 minutes) from the six students (mean number of submissions 6) related to the topic of study in the classroom was numeracy (e.g., exponents, factors, prime numbers, and square roots).

Daily mathematics lesson, as a secondary source of data, were also videotaped for the duration of the school year by a research assistant. This data assisted us in developing the prior context for each of the mathcam video diaries analyzed. The video data from the daily mathematics lesson was also a useful reference tool if clarification was needed in analyzing something said in the mathcam video diaries that were perhaps not altogether clear.

Data analysis

All video data were transcribed by two transcribers (i.e., the research assistant who did the classroom videotaping and a second research assistant). All coding was done twice to ensure reliability and consistency. The coding was done by the first author and the research assistant responsible for classroom videotaping.

Each mathcam video diary was coded by connected ideas or sentences within the excerpt to identify the following: (1) the distinctive feature of mathematical discourse (i.e., identifying uses of words that count as mathematical, use of uniquely mathematical visual mediators, use of discursive routines, and use of endorsed narrative) related to the excerpt, (2) the mathematical processes being utilized using those identified by the National Council of Teachers of Mathematics (2000) (e.g., reasoning and proof, problem-solving, communication, connection, and representation), and (3) the mathematical cognition demonstrated.

It should be noted that the interpretation of the mathematical processes may be highly subjective, and more than one mathematical process may be evident or applicable. The purpose in coding the excerpts with the final two codes is to demonstrate the way in which the excerpt and the distinctive feature of discourse are being interpreted as both evidence of mathematical cognition and of mathematical processes.

The primary analysis was aimed at ascertaining potential interactions between the four distinctive features of mathematical discourse used. Thus, these codes were examined both individually (i.e., the nature of the thinking occurring during each code) and in relation to one another (i.e., the nature of the thinking when one code was paired, or which pairings occurred or did not occur, etc.).

Also appropriated from Ben-Yehuda et al. (2005) is the use of *high-resolution descriptive methods* to report our data. This method is intended to focus on detailed qualitative accounts of students' mathematical thinking as evidenced in the mathcam video diaries through transcriptions and, in our case, video analysis. Accordingly, this research is qualitative.

It is important to note, that data drawn under these research conditions (i.e., self-reported thinking aloud during homework) versus other research conditions (i.e., experimental, interview, etc.) can be seen as a limitation of this research design in that it is a unique representation of mathematical cognition. However, comparison between child and researcher-observer reports in a study by Wu et al. (2008) investigating verbalized reports of cognition showed high consistency (Kappa = .948). Wu et al.'s results suggest that children can report on thinking accurately. Surprisingly, little research exists documenting students' mathematical cognition in naturalistic settings (i.e., while completing homework) (cf. Berger, 2004). The naturalistic approach to this data collection model (i.e., in the home) is a unique element to the research design.

We report in our Results and Discussion section primarily data related to our second goal for this research: to examine the interaction (or lack of interaction) between the four distinctive features of mathematical discourse proposed by Ben-Yehuda and colleagues in order to contemplate what potential interactions might suggest about students' underlying or emergent mathematical cognition/processes. In the Conclusions we discuss our perspectives on the first and third goals: to explore the potential utility of the proposed four distinctive features of mathematical discourse as an analytic tool in studying mathematical cognition and mathematical cognitive processes; to reflect upon implications of our findings to teaching, learning, and future research.

RESULTS AND DISCUSSION

Our results across all six students invited to participate in the mathcam video diaries portion of the research suggested that the there were particular forms of interaction between the four distinct features of mathematical discourse under examination: uses of words use of uniquely mathematical visual mediators, discursive routines, and endorsed narratives. Use of words, specifically those mathematical words used in the homework sheet from which the students referred to, created some source of difficulty for each of the six students during one or more of the mathcam video diaries analyzed. In all cases, the six students indicated in their mathcam video diaries that they would ask for assistance from their teacher. In one case, Kara, the student we will be discussing shortly, referred to the internet for further support.

Mathematical visual mediators were used very infrequently by the six students (n = 4) in an effort to support their mathematical cognition (see line 10). In all but one instance, the visual mediators used by the six students replicated those in the classroom by Duane during a lesson on integers (i.e., the use of colored counters to understand negative and positive numbers). Across all six students, an interesting interaction was observed between discursive routines and endorsed narratives. Instances where the six students were experiencing cognitive challenges, discursive routines were observed independent of endorsed narratives. We illustrate this interaction with two examples from mathcam video diaries submitted by Kara.

In the upcoming excerpt (lines 1to 13), Kara is describing a problem worked on during class. In this problem, the students were asked to explain why a square with an area of 20 cm^2 did not have a whole-number side length. They were also asked to consider which squares would have a whole-number side length.

Line	Excerpt	Feature of Mathematical Discourse	Mathematical Processes	Mathematical Cognition
1	So, today in math class we did the perfect square problem again, and while everybody else was doing the systematic trial system, I decided that I	Discursive routines	Problem-solving, Connection	A perfect square is a number that has a rational number as its square root.

	was gunna [SIC] try to figure out the perimeter, because if I could figure out the perimeter, I could divide it by four, and then I'd be able to find out the outside, then I could find out the width and the length for each side, and then from there I could multiply it and get the answer to the area.			
2	So, I was trying to figure it out, and I was trying to. . . first I made the rectangle, with the twenty square centimeters as the area, and then I cut off the four extra . . . um, squared centimeters that made it into a rectangle, and then I tried dividing that up.	Visual Discursive routine	Problem-solving, Representation	The geometric representation of a perfect square.
3	And though it would have taken me a long time to figure it out, I'm pretty sure it would have eventually worked. But my um, other members of my group found it quickly by us- well not quickly, but they found it eventually by using the systematic trial system,	Endorsed narratives	Communication	A perfect square is a number that has a rational number as its square root.

	but it took 'em [SIC] a while. Um, other people figured out a formula, and it turned out the entire thing was about roots, and I wish I would've thought about it because square root, it means the root of the square. If you think about it, it just kind a sounds like a big fancy word, but if you look at the word it makes sense to what it means.			
4	So if I thought about using the buttons on the calculator, maybe I could a figured it out. But, um, I definitely thought the way I decided to find the answer was kind of a cool way to try and do it.	Discursive routines	Connection	A perfect square is a number that has a rational number as its square root.

Instances where there were mathematical challenges exhibited by the six students, no interaction between discursive routines and endorsed narratives were observed. For example, Kara made inappropriate connections between her prior knowledge of finding the perimeter of a perfect square with square roots (line 2). She used a visual mediator to assist in her problem-solving to develop a representation of her thinking. However, her efforts did not move her forward in her trying to support her own mathematical cognitive development. Despite the fact that she was unsuccessful, she still viewed her approach as “cool” (line 4). She was unaware of her lack of endorsed narratives.

A major source of misconception for Kara was from prior knowledge from previous classroom instruction linking the side length of a perfect square with an area of 16 units and the root of 16, which are both four. In the upcoming excerpt (lines 5 to 13), Kara was solving the following problem from her homework sheet: *Estimate the value of each square root*. Even though she is only asked to estimate, Kara checked her estimations on her calculator. We see through her mathematical discourse that cognitively she was making inappropriate connections with the number four as an appropriate divisor for *all* perfect squares, rather than observing that the square root is only four when the number is 16. Although she recognized that her method was related to perimeter (line 7) and perhaps unrelated to square roots, she continued in her problem-solving with the strategy of dividing by four to find the remaining square roots.

Kara determined via her calculator that her method of estimation worked relatively well for several of the questions (lines 8 – 11), but failed her when trying to find the square root of 78. At this point, to justify her failure of reasoning and proof, she declared that her method of divide by four was not at issue but rather, according to Kara, the magnitude of the number under investigation (lines 12 – 13). The interesting point in this example was that the discursive routines that led her to incorrect mathematization were not simultaneously interacting with endorsed narratives. That is, Kara was engaging in the discursive routines without an understanding or awareness of the related endorsed narratives.

Line	Excerpt	Feature of Mathematical Discourse	Mathematical Processes	Mathematical Cognition
5	And right now I just have to find the square root, and I know how to find the square root quick because of the calculator I'm using. So, I just have to type it in. If I can find	Discursive routines	Problem-solving	A perfect square is a number that has a rational number as its square root.

	the button. So [inaudible] the square root of thirty six is . . . six!			
6	Now, number four is estimate the value of each square root. Explain how you found one estimate. So, twenty-three, [. . .]. Well, right now I'm just gunna [SIC] think of all the things that you can multiply twenty three by [. . .]. I think I'm gunna try to figure out what . . . five times four is . . . five point seven five. I think it's going to be... five point seven five... because that is what it is divided by four, and there's four. Wait that's perimeter!	Discursive routines	Reasoning and Proof	A perfect square is a number that has a rational number as its square root.
7	I always forget that it's not the perimeter! It's the inside of the square. When I tried doing this before, I figured out the perimeter before I figured it out but it didn't work very well.	Endorsed narratives	Connection	The geometric representation of a perfect square
8	I'm going to estimate. That it's going to be . . . I'm gunna [SIC] try it with thirty-six. Thirty- six divided by four is nine, the actual answer was three. So, if thirty-six is . . . if thirty-six, the perimeter for thirty-six is nine, but the	Discursive routines	Reasoning and Proof	A perfect square is a number that has a rational number as its square root.

	square root is six, then maybe for twenty three it'll be three less too.			
9	So, if I do . . . twenty-three divided by four again and add five point seven, maybe it'll be . . . I'm gunna [SIC] go with the number four. Wow the answer was four point eighty, so I was close!	Discursive routines	Reasoning and proof	A perfect square is a number that has a rational number as its square root.
10	It's definitely a good um, strategy for me to use if I just try and... if I just answer it that way then . . . I think I should try that for all my other questions too, and then I don't have to try and come up with some type of . . . I don't have to just guess a random number.	Discursive routines	Communication Connection	A perfect square is a number that has a rational number as its square root.
11	So thirteen divided by four is three point two five, [. . .]. I'm gunna [SIC] try three. The square root of three is my estimate. And the square root of thirteen is three point six one! See, my method definitely works, 'cause it might be over, but it's still close.	Discursive routines	Problem-solving, Reasoning and proof	A perfect square is a number that has a rational number as its square root.

12	Seventy-eight divided by four is 19.50, and I'm gunna [SIC] guess that it's gunna be a multiple of four and I'm not sure why but I'm going to subtract four from it and my answer's gunna be 15.50. So, my estimate is 15.50. And, the square root of 78 is eight point eighty three. Definitely didn't work out for me this time, but it's a good strategy. Probably because 78 is a larger number.	Discursive routines	Problem-solving, Reasoning and Proof	A perfect square is a number that has a rational number as its square root.
13	I'm going to explain how I got the answer for thirteen. I got my estimate by dividing the digit, no, the number . . . by dividing the number by . . . dividing 13 by four, number of sides, and then subtracting one because of . . . because it's the area and not the perimeter.	Discursive routines	Communication	A perfect square is a number that has a rational number as its square root.

In contrast to the preceding example, in the next set of transcriptions from another mathcam video diary, Kara described how she made sense of integers using the representations or visual techniques experienced earlier in the day in the classroom. She explained how Duane used red and blue counters to represent positive and negative integers and how this proved to be very useful in assisting her cognitive development (line 14). During this mathcam video diary Kara was working on the following question from the homework sheet: *A number line is used to add* *integers. Write the addition expression and the sum modeled by each diagram*. For each question, a mathematical statement is provided plus a number line.

In this next excerpt related to integers, discursive routines are interacting with endorsed narratives (lines 14 – 17). This interaction has two important outcomes. First, Kara is able to move forward in her development of mathematical cognition related to integers. Second, she expands her understanding using the combined discursive routines and endorsed narratives to hypothesize and make connections about other relationships between integers. She calls this a “strategy” (line 15), and then proceeds to engage in reasoning and proof to revisit her strategy to confirm her understanding (line 16). As she continued reading the homework sheet, she saw that the homework sheet ultimately outlined the “strategy” she had just developed (17). Her efforts are in contrast to those above (lines 1 – 4) where her reasoning falls apart and there are no endorsed narratives engaged to halt her misconceptions or redirect her learning.

Line	**Excerpt**	**Feature of mathematical discourse**	**Mathematical Processes**	**Mathematical Cognition**
14	So, today in math class, I really liked how Mr. Heidi explained integer-integers by using counters. He used white squares and red squares. And the red squares are positive and the white squares were negative, and what he did is he added and um, subtract them using integer numbers and it was really- it made it easier to understand when you could see that they	Words, Discursive routines, Visual, Endorsed narratives	Communication Representation	Addition and subtraction of integers.

	canceled each other out. Now, I'm on question three, and I've- I kind of do the same thing with the counters, except I do it in my head, which makes it- because I'm really good at that kind of mental math, it just kind of works out.			
15	So four positives, three negatives, it's three pairs, which makes it positive one. [. . .] So now I have a new strategy- now, when there is a higher positive number than the negative number that's getting added, from the negative number that's getting added, I can just subtract the positive number as if it was just both positive numbers. So ten subtract six, which is four. And I know the answers gunna be four, ‘cause there's four positives left. So I can just do that as a strategy now, so I don't always have to... so I don't always have to write it out.	Discursive routines, Endorsed narratives	Reasoning and proof, Problem-solving	Addition and subtraction of integers
16	I think the same thing might be for if there's a higher negative than positive, except it's the opposite- instead of	Discursive routines, Endorsed narratives	Reasoning and proof Problem-solving	Addition and subtraction of integers

	having the sum of the equation a positive number, the sum of the equation would be a negative number. So, seven, my prediction that seven minus two is five... so I believe the answer's gunna be negative five. Now, I'm just gunna prove my theory by doing one, two, three, four, five, six negatives and two positives. I'll circle the two positives, and that leaves one, two, three, four, five. That leaves five negatives left. That means that my hypothesis was correct.			
17	This is a second part to question number four. It says: when you add a positive integer and a negative integer, the sum is positive... when the numerical larger integer is. Oh! Positive, when the numerical larger integer is positive, negative when the numerical larger integer is negative... just like my theory- my strategy that I was doing when I was figuring out! And, the sum is going to be zero when the integers are the	Discursive routines, Endorsed narratives	Reasoning and proof, Connecting	Addition and subtraction of integers

	same number.			

The preceding example from Kara is important. It illustrates what was seen throughout the mathcam video diaries from all six students; namely, that the development of mathematical cognition, as shown by through the six students' mathematical discourse, was characterized by an interaction between discursive routines and endorsed narratives. Mathematical *mis*understanding was linked to inappropriately knowledge transfer from one context to another (i.e., Kara's use of 4 as the divisor and potential root of many numbers) and consistently lacked an interface with the endorsed narratives.

Our results do not suggest that the six students improved their understanding as a result of their verbalization of their mathematical cognition and mathematical cognitive processes (cf. Mercer et al., 1999; Sfard & Kieran, 2001). There was minimal evidence of the six students engaging in self-correction based upon their verbalization (n = 6). However, there were more instances (n = 12) where the six students verbalized incorrect calculations but proceeded with their work unaware of their errors. We hypothesize that in order for students to benefit from verbalization, discursive routines must interact with endorsed narratives simultaneously.

In summary, our research suggests development of mathematical cognition may be linked from a learning perspective to the extent to which students are able to connect discursive practices to endorsed narratives. Our findings do not suggest the same sort of requisite interactions for mathematical visual mediators and words. Rather, the latter two were seen as having supporting roles in students' emergent mathematical cognition.

CONCLUSIONS

In our research, we focused on the ways in which six students use distinctive features of mathematical discourse, as representations of mathematical cognition/processes, to support their own learning. Our results show that mathematical misconceptions were largely discursive routines that were not paired with endorsed narratives and thus the mathematization was often incorrect. Simply put, the six students participating in the mathcams video diaries portion of this research followed routines without sufficient understanding of the rules governing the routines. While this finding is not perhaps surprising, it is noteworthy given that completing homework inaccurately may reinforce mathematical misconceptions and given that homework completion is linked to greater achievement in mathematics (Cooper et al., 2006).

Our results raise three important pedagogical considerations for educators. First, it is of significant importance to examine when certain mathematical relationships do not hold or, in other words, fail to adhere to endorsed narratives of mathematics, such was the case in the square root problems that we outlined earlier. Duane used as his classroom example for determining square roots, the number sixteen – also represented as square with four equal sides of unit length four. The "divide by four" happen to work in the example he presented. A counter example, where "divide by four" was not an appropriate strategy finding a square root of a number, would have perhaps averted the misconceptions that plagued each of the six students with the square root problems during homework.

Second, students should also be explicitly encouraged and required to reflect upon endorsed narratives in mathematics (i.e., describing the rule, pattern, etc.). In some sense, over the last decade mathematics educators have been discouraged from approaching mathematics teaching and learning from "rules-based" perspective. Our research shows that a deep understanding of endorsed narratives is nevertheless necessary and thus may require some explicit instruction or explication.

Third, in supporting students who are experiencing challenges, educators should be encouraged to examine the extent to which students have an explicit understanding of endorsed narratives. Direct engagement with the student, examining the endorsed narratives they assume to be enacting may shed considerable light on the challenges a student may be having.

In our introduction, we make the case that the four features of mathematical discourse identified by Ben-Yehuda and colleagues (2005) are potentially useful analytic tools in analyzing mathematical cognition. One of our research goals was to explore the utility of the four distinctive features of discourse as tools to analyze mathematical cognition. Our results are promising in this respect. The pattern of evidence showing a lack of interaction between discursive routines and endorsed narratives within the data at instances of mathematical misconception suggest to us that using the distinctive features of discourse proposed by Ben-Yehuda and colleagues (2005) as an analytic tool can be a useful method of engaging in a fine-grained analysis of mathematical cognition/processes.

In conclusion, real-time data collection in naturalistic settings, utilizing modified talk- and think-aloud protocols (Ericsson & Simon, 1993, 1998) are strengths of this research. However, a limitation of this design is the inability to track more than six students at one time due to technological and financial constraints (i.e., only six laptops available for the project). Additionally, other researchers may identify other mathematical discourse frameworks that can also be useful in analyzing mathematical cognition. We make no claims that the methods used are uniquely suited to analyzing mathematical cognition, nor do we make the claims that conclusions we have presented can be generalized. More research is needed to determine the extent to which discursive routines in the absence of endorsed narratives lead to problematic mathematization. Additionally, further testing of the four features of mathematical discourse as an analytic tool is also necessary. Early results, however, are again promising.

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