REPRESENTATIONS IN MATEMATICS EDUCATION:

AN ONTO-SEMIOTIC APPROACH

Vicenç Font[1]
Universitat de Barcelona
Juan D. Godino
Universidad de Granada
Bruno D'Amore
Universidad de Bologna

ABSTRACT

Research in didactics of mathematics has shown the importance that representations have in teaching and learning processes as well as the complexity of factors related to them. Particularly, one of the central open questions that the use of representations poses is the nature and diversity of objects that carry out the role of representation and of the objects represented. The objective of this article is to show how the notion of semiotic function and mathematics ontology elaborated by the ontosemiotic approach of mathematics knowledge, enables us to face such a problem, by generalizing the notion of representation and by integrating different theoretical notions used to describe mathematics cognition.

Key words: external and internal representations, mathematical objects, meaning, understanding, semiotics

[1] vfont@ub.edu

1. IMPORTANCE AND COMPLEXITY OF REPRESENTATIONS IN RESEARCH IN MATHEMATICS EDUCATION

The large quantity of publications on the topic of representations, to which specific scientific meetings, monographs, conferences and journal publications have been dedicated (Janvier, 1987; Goldin, 1998; Cobb, Yackel y McClain, 2000; Goldin, 2002; Hitt, 2002; .....), show the importance of the same for mathematics education and at the same time its complexity. The reason for this interest should be found in the fact that to speak about representation is equivalent to speaking about knowledge, meaning, comprehension, modelling, etc. Without doubt these notions make up the central nucleus, not only of our discipline, but also of epistemology, psychology and other sciences and technologies that occupy human cognition, its nature, origin and development. This diversity of disciplines interested in representation is the reason for the diversity of approaches and ways of conceiving it.

The complexity implied in the use of representations is patent in the following quote of Goldin y Janvier (1998, p. 1), for whom the term "representation" and the expression "system of representation", in connection with mathematics teaching and learning, has the following interpretation:

> *"1. An external, structured physical situation, or structured set of situations in the physical environment, that can be described mathematically or seen as embodying mathematical ideas;*
>
> *2. A linguistic embodiment, or a system of language, where a problem is posed or mathematics is discussed, with emphasis on syntactic and semantic structural characteristics;*
>
> *3. A formal mathematical construct, or a system of construct, that can represent situations through symbols or through a system of symbols, usually obeying certain axioms or conforming to precise definitions – including mathematical constructs that may represent aspects of other mathematical constructs;*
>
> *4. An internal, individual cognitive configuration, or a complex system of such configurations, inferred from behaviour or introspection, describing some aspects of the processes of mathematical thinking and problem solving".*

These different uses of the notion of representation show the different components and facets implied in mathematics activity, the situations in which language and personal and cultural objects arising from this activity, are developed. In our opinion, the complexity of the topic, the ambiguity of the representations and their importance are in the mathematics objects which are trying to be represented, their diversity and nature. To speak about representation (meaning and comprehension) necessarily implies speaking about mathematics knowledge, and so, mathematics activity, its cultural and cognitive "productions" and also those related to the world which surrounds us.

Representation is characterised using abstract correspondence between two entities which are put in some referential relation one against another by an actor or an observer, "that deliberately pays no attention to what kinds of things are involved in the correspondence" (Kaput, 1998, p. 266). We consider it necessary to say what is being represented and in what way, since this lack of explanation can imply bias in the style of description and the adoption of hypothesis about what is cognisable and the ways of knowing. One "ingenuous" reply given to this problem is to say that mathematic concepts are being represented. But, what are mathematics concepts? Furthermore, concepts are not the only constituents of mathematics knowledge: we also find, problems, notations, procedures, propositions, arguments; systems or mathematics structures, theories. All these things we refer to are "mathematic objects" (they intervene in the mathematical activity) and also have to be represented and understood. Even the "material representations" themselves are frequently represented by each other. One aspect that increases the difficulty of the problem is that we often use the same term for different things. With the expression "real number" for example, we refer to a concept (a rule that enables us to recognise an object) as well as to a whole structure or mathematics system.

In this paper we will look at the "ontological problem" of the representations and other related questions from the holistic approach which proposes the ontosemiotic approach of cognition and mathematics instruction (Godino and Batanero, 1998; Godino, 2002; Godino, Batanero and Roa, 2005; Contreras, Font, Luque and Ordóñez, 2005). The notion of semiotic function and the ontology that proposes this theoretical approach, generalises and clarifies in a radical way the notion of representation and provides a solution to the aforementioned ontological problem. To be more precise, we are going to deal with the following problematic aspects of the representations:

1. The nature of the objects that intervene in the representations.

2. The distinction between internal and external representations.

3. The problem of the representation of the generic element.

4. The role that the representations of one object plays.

5. Processes of comprehension and its relation with the translations between different representations.

We have organised the paper in the following sections:

- In the first section the problem and the objectives of the paper are posed.
- In the second we briefly present the theoretical framework of the ontosemiotic approach showing the solution proposed to the ontological problem of representation and meaning.
- In the third section we study the problem that internal and external classification presents.
- In the fourth section we reflect on the role of "generic element" in mathematics and its relation to the representations.
- In the fifth section we study the problem of considering that there is one same mathematical object that has multiple different representations.
- In the sixth section we reflect on the problem of comprehension and its relation with the translation between different representations.
- Finally, in the seventh section, we present a synthesis of the response given by the ontosemiotic approach to the questions posed and we end with some general conclusions.

In the Annex we present an episode of class that we are going to use as a context of reflection and to show the type of application that we do to the representation topic of the theoretical constructs elaborated by the ontosemiotic approach. This involves responses given by four secondary school students (17 years old) to an item of a questionnaire proposed in the study process of the derivative.

2. THE ONTOLOGICAL PROBLEM OF REPRESENTATIONS AND MEANING

The use of the terms “representation” and “meaning” is carried out in circumstances in which an entity, frequently a linguistic entity starts a relation with another. This use is similar, on one hand to the mathematics relation when a function between mathematics objects is defined (correspondence) and on the other hand to the use made in ecology, when an “object” fulfils and plays a role.

This relational aspect of representation and meaning does not seem conflictive. The problem arises when we are interested in the types of objects that are related, the correspondence criteria and the finality with which the relations are established. The conflict arises when in addition to language and the objects of the world which surrounds us, non ostensive entities that we usually designate as concepts, notions, ideas, abstractions, come into play. These entities are always present in our mind and in any communicative act.

Vergnaud (1990) tried a clarifying contribution and joined in the notion of concept its same constructive component. For Vergnaud the decisive point in the conceptualization is the step of the concepts-like-instrument to the concepts-like-objects and an essential linguistic operation in this transformation is the nomination; this author understands “conceptualisation” as “conscious appropriation” when proposing the triplet (S, I, *S*), where S is the referent, I the meaning and *S* the signifier, as the definition of a concept. Vergnaud´s idea could be considered as a continuation to the classic line, the one that passes through the three famous “triangles” (D’Amore, 2006):

- Peirce´s triangle: interpretant, representamen, object.
- Frege´s triangle: sense, expression, reference.
- Ogden and Richards´s triangle (1923): reference, symbol, referent.

Steinbring (1997) also contributes a clarifying interpretation of the epistemological triangle when he proposes that the “”meanings of the mathematical concepts emerges in the game between the signs/symbols and the contexts of references o domains of the objects” (p.50).

Each one of these proposals can be seen as the description of the relation between a language that "describes a world" and "the conceptual objects of the world". If one of these worlds is mathematical, things become more complicated because in mathematics we find objects of different nature:

- Objects that "compose" the world – the mathematics world- and that are represented by systems of signs (point, number, plans,….)
- Objects that relate the objects of the said world (relations, like equality; operations, like algorithms,…)
- Objects that describe the presence of the previous objects in more complex situations that characterize the mathematics world (problems, demonstrations,...)

In different papers Godino and collaborators (Godino and Batanero, 1998; Godino 2002; Contreras, Font, Luque, Ordóñez, 2005; Godino, Batanero and Roa, 2005) have developed a set of theoretical notions that form an ontological and semiotic approach of cognition and mathematics instruction, due to the central role that they assign to language, to the processes of communication and interpretation of the variety of intervening objects. We will use the expression "ontosemiotic approach" to refer to this way of approaching didactics of mathematics".

The ontosemiotic approach of mathematics cognition has faced the problem of meaning and representation by elaborating an explicit mathematical ontology based on anthropological (Bloor, 1983; Chevallard, 1992), semiotic and sociocultural (Sfard, 2000; Radford, 2003; Ernest, 1998) assumptions. This supposes assuming certain socioepistemic relativity for mathematical knowledge since knowledge is considered to be indissolubly linked to the activity in which the subject is implied and is dependent on the cultural institution and the social context which it forms part (Radford, 1997). We attribute an essential role to language, in its double representational and instrumental valence, in the constitution of mathematical objects. The discursive mathematical activity itself including its continuous production of symbols, "creates the need for mathematical objects; and these are mathematical objects (or rather the object-mediated use of symbols) that, in turn, influence the discourse and push it into new directions" (Sfard, 2000, p. 47).

As a result of this and in consonance with the proposal of symbolic interactionism (Blumer, 1969) we adopt a radically more general and more committed conception of *object* than the definition given by Aristotles in his *Metaphysics*. For us "mathematics object" is any entity or thing which we refer to, or that we speak about, be it real, imaginary or of any other type, that intervenes in any way in the mathematics activity. Aristotle (in Metaphysics) defines the "thing", when he starts from real, attributing three characteristics to it: (1) three dimensionality, (2) accessibility (3) possibility to separate material from other parts of reality from other "things".

To follow we are going to synthesize the ontology proposed in the ontosemiotic approach of mathematics cognition.

2. 1. Systems of operative and discursive practices linked to fields or types of problems

All kinds of performances or expressions (verbal, graphic, etc.), carried out by someone in order to solve mathematics problems, communicate the solution obtained to others, validate it or generalise it to other contexts and problems, are considered to be *mathematics practice* (Godino y Batanero, 1998). In the study of mathematics, rather than a particular practice to solve a specific problem, it is interesting to consider the systems of practices (operative and discursive) carried out by people when faced with problematic types of situations. To the questions, what is the mathematics object arithmetical average?, or what does the expression "arithmetic average" mean? The reply, "the system of practices that a person carries out (personal meaning) or when shared institutionally (institutional meaning) to solve a type of situation-problems in which it is necessary to find a representative value of a set of data, is proposed.

2.2. Intervening and emerging objects of the system of practices

In mathematics practices ostensive objects (symbols, graphs etc.) and non ostensive objects (which we bring to mind when doing mathematics) which are textually, orally, graphically or even gesturally represented, intervene. New objects that come from the system of practices and explain their organization and structure (types of problems, actions, definitions, properties, arguments), emerge. If the system of practices are shared in the heart of an institution, the emerging objects are considered to be "institutional objects whilst if these systems correspond to a person we consider them as "personal objects".

2.3. Relations between objects: Semiotic function

Hjemslev´s (1961) notion of the function of sign[2] as the dependence between a text and its components and between these components themselves, is adopted. So, we are dealing with the correspondences (relations of dependence or function) between a antecedent (expression, signifier, representant) and a consequent (content or meaning, signified), established by a subject (person or institution) according to a certain criteria or corresponding code. These codes can be rules (habits, agreements) that inform the subjects implied about the terms that should be put in correspondence in the fixed circumstances.

For us, the relations of dependence between expression and content can be representational (one object which is put in place of another for a certain purpose), instrumental (an object uses another or others as an instrument) and structural (two or more objects make up a system from which new objects emerge). In this way semiotic functions and the associated mathematics ontology, take into account the essentially relational nature of mathematics and radically generalize the notion of representation. The role of representation is not totally undertaken by language: in accordance with Peirce´s semiotic, it is proposed that the different types of objects (situations-problems, actions, concepts, properties and arguments), can also be expression or content of the semiotic functions.

2.4. Configuration of objects

The notion of "system of practices" is useful for certain types of macrodidactic analysis, particularly when dealing with comparing the particular way mathematical

[2] Named by Eco (1979) as semiotic function.

knowledge adopt in different institutional frameworks, contexts of use or language games. For a finer mathematics activity it is necessary to introduce the six types of primary entities: situations, actions, language, concepts, properties and arguments. In each case, these objects will be related among themselves forming "configurations", defined as the network of emerging and intervening objects of the systems of practices and the relation established between them. These configurations can be epistemic (networks of institutional objects) or cognitive (network of personal objects). The systems of practices and the configurations are proposed as theoretical tools to describe the mathematical knowledge, in its double personal and institutional version.

2.5. Cognitive dualities

The notion of language game (Wittgenstein 1953) occupies an important place when considered together with the notion of institution, like the contextual elements that relativize the meanings of the mathematical objects and attribute a functional nature to them. The mathematics objects that intervene in mathematics practices and those emerging from the same, depending on the language game they are taking part in, can be considered from the following facets or dual dimensions: personal-institutional, elemental-systemic, expression-content, ostensive-non ostensive and extensive-intensive (Godino, 2002)**.** These facets are grouped in pairs that are dually and dialectically complemented. They are considered as attributes applicable to the different primary and secondary objects, giving rise to different "versions" of the said objects.

In Godino, Batanero and Roa (2005) the six types of primary entities and the five types of cognitive dualities are described using examples from a research in the field of combinatory reasoning.

The types of objects described, summarised in figure 1 (systems of practices, emerging entities, configurations or ontosemiotic networks, the cognitive dualities or contextual attributes, together with the notion of semiotic function as the basic relational entity) make up an operative response to the ontological problem of representation and meaning of mathematical knowledge.

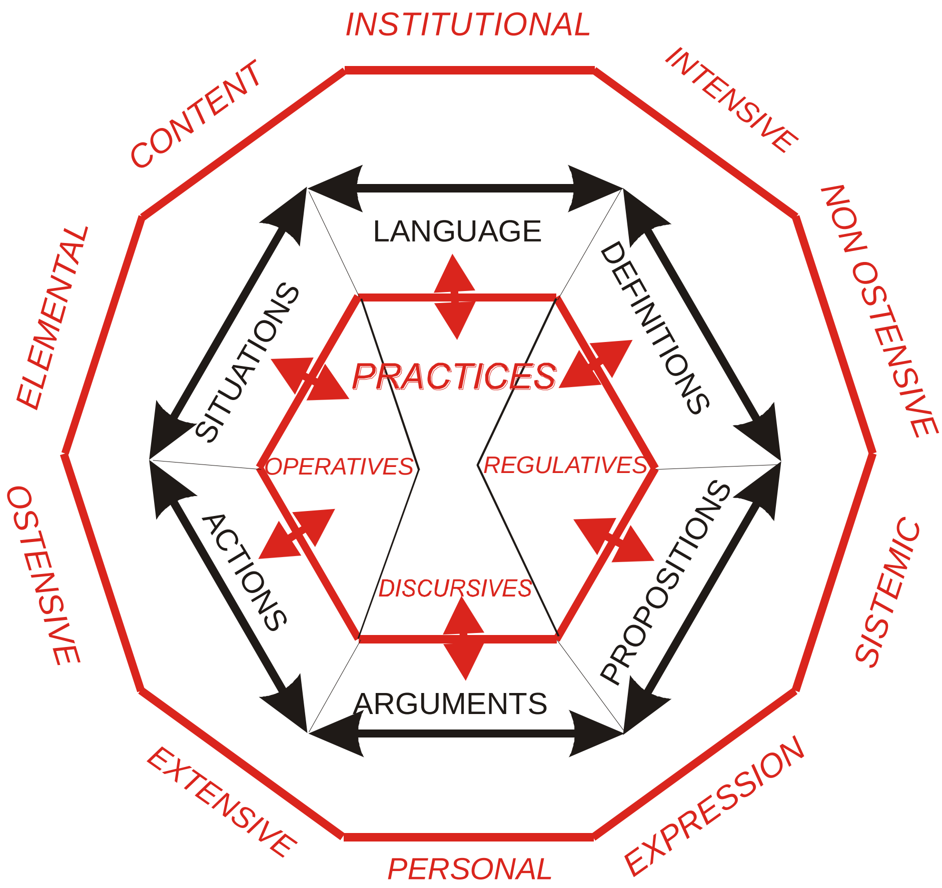


Figure 1: Ontosemiotics of mathematical knowledge

In the following sections we will show how the five dimensions or cognitive dualities, as well as the other theoretical instruments elaborated by the ontosemiotics approach and in particular the notion of semiotic function enable us to face the complexity that research on representation requires. Furthermore, we will try to relate these facets with different problematic aspects of the representations that other authors have dealt with.

3. THE PROBLEM OF THE CLASSIFICATION OF REPRESENTATIONS IN INTERNALS AND EXTERNALS

In the different research programmes in mathematics education we can find among others, two uses of the term representation. On one hand, this term is used to describe the cognition of the people; in this case it usually is accompanied by the term "mental" or "internal". On the other hand, it is normal to use the term representation when referring to the systems of public signs (ostensive) that are the most important tools for the mathematics activity. When the term "representation" is used with this aim it is usually accompanied by the term "external".

In the example described in the Annex, it is obvious that on one hand there are representations that literature describes as external (graphs, symbolic expressions etc.). On the other hand, the difference between the students´ responses enables us to suppose that there are representations, usually considered internal that are related to the different responses of each student.

This first classification in mental or internal representations and external representations is not by any means a transparent classification. The ambiguity of the internal/ external classification has been pointed out by different researchers. For example Kaput in relation to this question asks: "What is it [a mental representation]? What do we mean when we say it "represents" something? For whom? How? What is the difference between the experience of an internal representation and that of an external representation? And is an external representation a socially or a personally constituted system?" (Kaput, 1998, p. 267).

Furthermore the classification between internal and external representations obliges us to ask what goes first, the internal before the external or vice versa. The majority of the cognitive psychologists consider the internal representations to be more basic since they consider that for the external representations to be representations they have to be mentally represented by their users and on the other hand, the mental representations can exist without a public duplicate; for example, many of our memories are never communicated. To the contrary, the majority of the social scientists and many philosophers inspired by Wittgenstein, do not agree with this.

Wittgenstein with his critics of the notion of "sensation" as "a private object of conscience" and his reflections about the relation between thought and language set the basis for the critic from the point of view that considers that external representations are the instruments with which we exteriorize our internal representations to make them accesible to other people. Wittgenstein questions the dualist vision that considers a mental world different from the material world and that admits the existence of an internal experience in which we grasp "interior" events. In his critic of the point of view that thought and language are two different types of processes inter-related by a causal link, proposes the following alternative: to think and to speak are one skill that can be presented linguistically or not linguistically. This opinion is expressed in paragraphs like the one to follow: "We could also, occupied in certain measurements, act in such a way that whoever should see me would say that I had thought – without words - : If two magnitudes are the same as a third one, they are equal among themselves.- But what is considered here as thought is not a process which has to accompany words in order to be pronounced without thought" (Philosophical Investigations, 1953, ¶ 330).

In the ontosemiotic approach the internal/external classification, in addition to be problematic, it is considered as not very operative and so it is proposed to convert it into two dualities or contextual attributes that in our opinion are more useful. We are referring to the ostensive-not ostensive and personal-institutional dualities.

We consider that the internal/external duality does not explain the institutional dimension of mathematics knowledge, thus confusing to a certain extent the said objects with the ostensive resources that are used as support for the creation or emergency of the institutional entities. This has serious consequences to understand the learning processes, since the role of human activity and the social interaction are not adequately modelled in the production of mathematics knowledge and in learning.

To follow we are going to show the relevance that considering the personal-institutional facet has when analysing the students´ responses. The analysis of the students´ four responses to section c of the questionnaire (Annex), permits us to suppose relevant differences between the mental processes that "occur" in the mind of each one of them. At this point, in accordance with the holistic point of view, when we are talking about the representations that we have proposed in section 1 of this article, we consider it necessary to take into account the personal-institutional duality before first reflecting on their mental processes.

It is not enough to reflect on the cognitive processes that have permitted these four students to reply to the questions in the questionnaire by carrying out a conversion from a graphic representation to a symbolic representation when they still did not know what the derived function of the exponential function of base *e* was. It is necessary to take into account, above all, the process of instruction that these students have followed if we wish to give an explanation of the achieved learning.

The analysis of the four responses, when it detects important differences between them enables us to observe that the four students apply the same type of practice to calculate the derivative of the function $f\,(x) = e^x$. The technique used consists of considering, to start with, a particular point with the tangent drawn (and so its abscissa and ordinate, are not considered to be variables). To follow, from the manipulation with dynamic computer programmes, like *Cabrigéomètre* or *Calcula*, we find a condition that fulfils all the tangent straight lines (in this case when the subtangent is always a segment of length 1), and this permits the calculation of its slope. Finally, students have to be clear that the condition they have found, and the calculation of the slope from which it is derived is valid for any point, so the point which was initially considered as a particular point is then considered as any point.

In order to answer this questionnaire, as well as using the graph of the function the following symbolic expression of the graph, $f(x) = e^x$, should be used. So, this technique relates the following ostensive points:

Graph of $f\,(x)$ and symbolic expression of $f\,(x) \Rightarrow$ Symbolic expression $f\,'(x)$

With this scheme, we symbolise that the student´ s starting point of the actions to find a condition that fulfils all the tangents, is the graph of the function. The symbolic expression of $f\,(x)$ is necessary to symbolise the condition that fulfils all the slopes of the tangent straight lines, which enables us to deduce the symbolic expression of $f\,'(x)$. If the students have practiced the calculation of the slope and the geometric meaning of the derivate of a point, they may obtain the symbolic expression of $f\,'(x)$ without much difficulty.

This method has a limited application field since previously the student has to discover a property that fulfils all the straight line tangents. So, this can also be applied to the family of functions that have a graph that is a straight line and also to the exponential and logarithmic families. The fact that this method can be applied to calculate the derivate of the exponential and logarithmic functions permits an organization of the didactic unit on derivates that have important curricular consequences (for example, that permits not to have to study the indetermination 1^{∞}).

One of the relevant aspects to frame the relation between the representations of the task of the questionnaire in the process of instruction in which it was carried out is that it enables us to know that in this instruction process the teacher opted to include for the institutional intended meaning, and also to the implemented and the evaluated neanings, practices that form part of the historic-epistemological evolution of the derivative object and which usually they are not contemplated in the didactic units about the derivative that can be seen in the curriculum of different countries.

4. THE PROBLEM OF THE REPRESENTATION OF THE GENERIC ELEMENT

One of the crucial characteristics of mathematics activity is the use of generic elements that is, a set or system of elements in one unit. This practice can be useful in the definition processes; for example a rational number is a class of ordered pairs of whole numbers that satisfy a relation; the generic element (a, b) $\in$ Zx(Z – {0} is non other than the scheme which includes many pairs of the same class for example, [(1,2), (5, 10), (3, 6), ...], thought of one same act of thought. At other times the generic element is useful for an economy of thought: for example, the fact that the three heights of a triangle come together at the same point does not depend on the type of triangle we are talking about, so any try to demonstrate this should refer to a scheme of possible triangles and not one specific one.

However, a dialectics between the generic element and the general element that frequently cause a greater cognitive complexity, arise. The mathematics reasoning, to go from the general to general, introduces an intermediate phase that consists of contemplating an individual object. This fact poses a serious dilemma: if reasoning has to be applied to a specific object, it is necessary for there to be some guarantee that we reason about any object so that it is possible to justify the generalisation in which reasoning ends. Furthermore, since the specific object is associated with its representation the problem of whether the representation is of a specific object or a general concept, appears (D'Amore 2005).

The introduction of the extensive/intensive duality in the ontosemiotic approach can help to clarify the problem of the use of generic elements (Contreras, Font, Luque and Ordóñez, 2005). Three questions which are different but connected have to be considered with respect to this problem:

1) Why does an intermediate phase that refers to a specific object intervene in the demonstration of a mathematical proposition (the statement of a definition, etc.)?
2) How is it possible that a reasoning in which a similar intermediate phase, in spite of this, gives rise to a universal conclusion?
3) The particular element normally forms part of a chain in which the previous links are generic elements. At the same time, the particular element to be considered as generic will be converted into the previous link of a new particular case and so on.

The extensive/intensive facet becomes as essential instrument to analyse the complexity associated with these three aspects. Expressed differently, the use of the generic element is associated with a complex net of semiotic functions (and so representations) that relate intensive with extensive objects. We will show this with the example of the students´ responses included in Annex.

If we observe the three sections of the questionnaire (Annex) we can sense that in the statement the step from the particular to the general has been taken into account. In section *a* students are asked to calculate the derivative for the three specific values (0, 1 and 2). In section *b* they are asked to calculate the derivative for a specific value "a" and in section *c* for any value. That is, the change from extensive to intensive has been present in the design of the questionnaire. In this process, we can observe that the extensive objects "represent" the intensive ones.

If we consider the technique of the derivative calculation that has to be applied in the questionnaire, and without going into a detailed analysis like those done in Contreras, Font, Luque and Ordóñez (2005), the following net of semiotic functions can be identified.

In order to calculate the derivative function from a condition that fulfils all the tangents, the student has to:

1) Treat the variables related by the formula and the graph of the exponential base *e,* separately. To do this, it is necessary to understand the exponential function of base *e* as a process in which other objects, one being *x* and the other being *f(x)*, intervene. Here, a semiotic function that relates the object *f(x)* to the object *x,* is established.

2) Associate *x* to the slope of the tangent straight line on the point of the *x* axis. This relation can be considered as a semiotic function that relates the object *x* with the object "slope of the tangent straight line on the point of the *x* axis."

3) Associate the expression that permits us to calculate the slope of the tangent straight line on the point of the *x* axis with *f´(x*). In this case we have a semiotic function that relates one notation with another different but equivalent one.

4) Consider *x* as a variable. In this case we have a semiotic function that relates an object to the class it belongs to.

5) Understand the function obtained as a particular case of the "derivative function" class. In this case we have a semiotic function that relates an object to the class it belongs to.

If we look at the questionnaire handed to the students we can observe that the sequence of sections aims at making the establishment of these semiotic functions, easier. The use of the letter *a* and of equality *x = a,* in section b of the questionnaire they have the role of introducing a specific element in the student´s reasoning and so make step 1, easier. The option to use the graph and the formula and the use of symbolic notation together for the coordinate point (a, e^a) is because they want the student to carry out steps 2 and 3. Steps 4 and 5 are intended to be achieved from the question in section *c.*

This example permits us to illustrate a phenomena that we consider to be very relevant: *the student in order to carry out the majority of mathematical practices, has to activate a net of complex semiotic functions and the ostensive used are determinant, both to reduce or increase the complexity of this net or to carry out the practice correctly.* For example, if we would have eliminated section *b* in the questionnaire, we would still want the student to apply the technique to calculate the derivative function and we would still use graphs (the ones of the previous activity with the computer and those belonging to section *a*) and symbolic expressions (section *c*), however the complexity of the semiotic functions that the student would have to carry out, would increase considerably and so also the possibilities of solving the task.

On the other hand, we should point out another factor which is just as or even more important than the representation used, that intervenes in a determinant way in the complexity of the net of semiotic functions associated with the use of generic elements: the rules of the language game in which we are situated. When we use an ostensive in mathematics practices as a generic element we are acting on a specific object, but we situate ourselves in a "language game" in which it is understood that we are interested in its general characteristics and we disregard the particular aspects. The analysis of dialogues between teachers and students related to the use of generic elements (for example those mentioned in Contreras, Font, Luque and Ordoñez (2005)), is necessary to know the details about the characteristics of this language game and of the difficulties that the students have to take part in it. The assimilation of the rules (or not) of this language game is fundamental to make up the net of semiotic functions associated with the practices in which the generic element intervenes.

5. THE PROBLEM OF THE MULTIPLE REPRESENTATIONS OF ONE "SAME" MATHEMATICS OBJECT

Frequently we say that one same mathematics object (function, derivative,....) is only given by certain representations (algebraic, graphs, tables,....). We think that this way of conceiving the role of the representations in the mathematics work and in the conceptualisation processes is a little "ingenuous".

It is enough to look with a historic perspective at any mathematics object to illustrate the complexity of the relations that are established between a mathematics object, its associated ostensives and the situations in which the object is used (in addition to the ostensives and associated practices) to organise phenomena. We take the cisoid as an example and we consider it defined[3] as a geometrical locus in the framework of the synthetic geometry. Within this research program, the definition of the cisoid enables us to represent it by the drawing of a curve. In fact, in the construction of figure 2, carried out by the Cabri Géomètre programme, the cisoid is represented by tracing point *P* when moving point *M.*

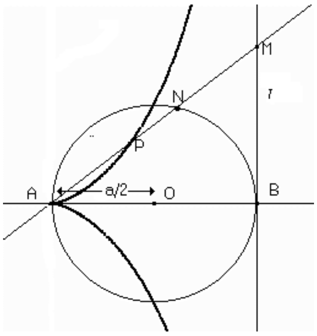


Figure 2

If we situate ourselves in the framework of analytic geometry and we use analogue techniques to those used by Descartes in *The Geometry* we can easily obtain the following *representation* of the cisoid: $x^3 + y^2x - ay^2 = 0$. This translation "Curve $\Rightarrow$ symbolic equation" is a technique that does not live alone but needs a theoretical body that justifies and allows it to make sense.

The research programme, initiated by Descartes, is a global programme in which local study is not considered. While we limit ourselves to look for an implicit expression we are moving from a global point of view. However, when we consider obtaining the explicit expression of the cisoid we are obliged to introduce local reasoning. When situated in this new research programme (local perspective), the development techniques in power series enable us to obtain explicit expressions of the cisoid. A historical look also shows that the different ostensive forms that can represent a mathematical object are the result of a long evolution, which, in some cases a new form of representation give expression to a new research programme.

[3] Let *C* be a circumference with a radius *a/2* and centre *O*, *AB* a diameter of *C* and *l* the straight line tangent to *C* in *B*. For each straight line AM, $M \in l$, we consider its intersection *N* with *C* and a segment *AP*, $P \in AM$, of the same length as *MN.* The geometric locus of the points *P* obtained is a curve called *Diocles´ Cisoid.*

For mathematics education we consider that it is important to show the ingenuity of the point of view that considers the ostensive representations of the mathematical institutions simply as different meanings of the same object. This consideration tends to underestimate the importance of the different ostensive representations, the configuration of the objects considered and the translations among them in the production of the global meaning of the said object (Wilhelmi, Godino y Lacasta, 2005).

The fact that the ostensive representations are framed in research programmes and that they imply the use of configurations or complex ontosemiotic networks, has serious implications. To follow we will indicate three of the most important.

(1) The first is that the representations cannot be understood on their own. An equation or a specific formula, a specific regulation of multibase blocks, a particular graph in a Carthesian system only acquire meaning as part of a larger *system* with established meanings and conventions. “The representational system important to mathematics and its learning have **structure**, so that different representations within a system are richly related to one another” (Goldin y Stheingold, 2001, p.1-2). So, it is more convenient to speak about epistemic configuration (or cognitive, if we refer to personal systems of practices) rather than to speak about ostensive representations or signs in order to make clear the net of objects and relations that are in play when the semiotic register or the context of use is changed.

(2) The second is the fact that the same object can be classified in two different research programmes, each one with their systems of representations, implies that “each representation” can be converted into “represented object” of the representation of the other research program. When the cisoid is studied in the framework of analytic geometry, a complex net of semiotic functions whose beginning and end can be represented by figure 3, is activated:

EXPRESIÓN	CONTENIDO
	$x^3 + y^2x - ay^2 = 0$

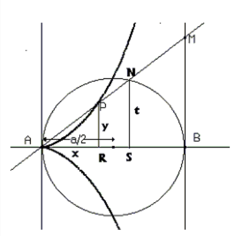
EXPRESIÓN	CONTENIDO
	CISOIDE

Figure 3

So, depending on the context, the curve can provide a geometric representation of the equation, or the equation can provide an algebraic symbolism of the curve. This fact leads us to consider that the cisoid can be represented by a curve in "synthetic geometry" and by an equation in "analytic geometry".

(3) The third is that an ostensive representation, on one hand has a representational value: it is something that can be put in place of something different to itself and on the other hand, it has an instrumental value: it permits to carry out specific practices that with another type of representation would not be possible. The representational aspect leads us to understand representation in an elemental way "something" for "something". However, the instrumental value leads us to understand representation in a systematic way, like an "iceberg" of a complex system of practices that the said representation makes possible.

In the ontosemiotic approach, the introduction of the elemental-systemic duality in the analysis of the representations enables us to reformulate the "ingenuous" vision that "there is one same object with different representations". What there is, is a complex system of practices in which each one of the different pairs object/representation (without segregating them) makes possible a subset of the set of practices that are considered the meaning of the object. Expressed differently, the object considered as emergent from a system of practices can be considered as unique and with a holistic meaning. However, in each subset of practices the object/representation pair (without segregating) is different, in the sense that it makes different practices possible.

In the example of the questionnaire (Annex), the use of graphic representation in a dynamic software is necessary to find a condition that fulfils all the tangents (the starting point of the questionnaire). In order to answer approximately section *a*, all you need is the graphic representation, however to answer exactly it is necessary to use the symbolic expression of the exponential function, too. To answer section *b,* it is necessary to use the subset of the graphic and symbolic representation. Expressed differently, the technique that the school institution intends the students to apply in this questionnaire is only possible if the graphic and the symbolic representations are introduced at the same time. If the graphic representation is not contemplated, the technique is viable. Contemplating the graphic representation, in addition to the symbolic representation, enables us to carry out specific practices that with the symbolic representation alone would not be possible.

6. THE COMPREHENSION PROBLEM AND ITS RELATION TO THE TRANSLATION BETWEEN DIFFERENT REPRESENTATIONS

From the cognitive point of view the comprehension of a mathematics object is understood basically in terms of integration of mental representations. This integration is the one that assures competence in the use of external representations associated with the object. From this perspective, a central objective in mathematics teaching consists of making the students capable of changing from one representation to another, but it is recognised that this objective is difficult to achieve. “The conversion of representations is a crucial problem when learning mathematics” Duval (2002, p.318).

The crucial importance given to the conversion of representation (in teaching and in learning) from a cognitive point of view is the result of understanding the “comprehension” basically in terms of mental processes. In the ontosemiotic approach of mathematics cognition we recognise the important role that the different ostensive representations and translations among them, play in mathematics comprehension (Font 2001; Font and Peraire 2001), however we adopt an anthropological and pragmatist view about this issue.

From a pragmatic point of view “to understand” or “to know” a mathematics object consists of being capable of recognising its properties and characteristic representations, relate it to the remaining mathematics objects and use this object in the whole variety of prototypical problematic situations which are proposed to them in the classroom. From this point of view the comprehension reached by a subject at one particular time will hardly ever be total or null rather it will be partial and progressive.

From this perspective, the teaching processes should be understood as the presentation of sequences of activities that they have as an objective, in time and with the resources available, the emergency of personal mathematical objects whose meaning is better adapted to the institutional meaning implemented by the teacher. It is understood that a student has understood a specific content when he uses it competently in different practices. So, comprehension is basically understood as a capacity that the student has and not so much a mental process that is produced in his mind when he/she articulates representations. The capacity is translated into practices that are publicly assessable, whilst the mental process is a private experience of the person.

When the meaning of a mathematical object is defined in terms of practices, the meaning becomes linked to other meanings and other objects, since in the practices this object intervenes with other mathematical objects. This fact enables us to distinguish two terms that are difficult to distinguish, we are referring to sense and meaning. In fact, since the object can be related to one or other objects according to the context, the type of notation, etc. in order to give rise to different practices, we can understand the sense as a subset of the systems of practices. The meaning of a mathematical object is understood as a system of practices that can be divided into different classes of more specific practices used in a specific context and with a certain type of notation thus producing a specific meaning. Each context helps to produce meaning (it enables us to generate a subset of practices), but it does not produce all the senses (Wilhelmi, Godino and Lacasta, 2005).

From an ontosemiotic point of view, the problem is not whether it is necessary to introduce only one representation of an object or more than one, nor what translations or relations between representations should be taken into account. The problem is really in determining whether the representations introduced facilitate or not the realisation of the practices that form part of the overall meaning of the student, in knowing whether the semiotic complexity increase or decrease and also in knowing whether unnecessary semiotic conflicts are produced or not.

7. SYNTESIS AND CONCLUSIONS

In this paper we have described some problematic aspects of the use of the representations in mathematics education and we have given a reply from the theoretical framework that we name the ontosemiotic approach.

With respect to the problem of the distinction between internal and external representations, in the ontosemiotic approach, this classification in addition to being problematic, is considered not to be very operative and so we propose to reconvert it into two facets which in our opinion, are more useful. We refer to the ostensive-not ostensive and personal-institutional facets.

With respect to the problem of representation of abstract entities we propose to analyse it in terms of cognitive duality extensive-intensive. When we use an ostensive as a generic element in mathematics practices we are acting on a particular object, but we situate ourselves in a “language game” in which it is considered that when we refer to this particular object, it is understood that we are interested in its general characteristics and we disregard the particular aspects.

With respect to the problem of whether there is one same mathematics object that has different multiple representations, this point of view is “ingenuous” for the ontosemiotic approach. The introduction of the elemental-systemic duality in the analysis of the representations permits the reformulation of this vision in the following way: What there is, is a complex system of practices in which each one of the different object/representation pairs (without segregation) permits a subset of practices of the set of practices that are considered the meaning of the object. Expressed differently, the object considered as emergent of a system of practices can be considered as unique and with a holistic meaning. However, in each subset of practices, the object/representation pair (without segregation) is different, in that it makes different practices possible.

With respect to the comprehension problem and its relation to the translation between different representations, it is argued that from an ontosemiotic point of view, the problem is not whether we must introduce only one representation of an object or more than one, or what translations or relations between representation have to be taken into account. The problem is really in determining whether the representations introduced make the realisation of the practice easier or not, in knowing whether the semiotic complexity increases or decreases and also in knowing whether unnecessary semiotic conflicts are produced or not.

As general conclusions of this work, the first that we want to point out is that an ontosemiotic approach to the representation and meaning, like the one we have described in this paper, is a holistic glance of the same, which permits facing up to the great complexity associated with the use of these notions in mathematics education. This holistic glance helps to understand the phenomena of the representation and the meaning as the visible part of the "complex iceberg" in the base of which we find ourselves with a net of objects, practices and associated ostensives, structured in epistemic (and cognitive) configurations.

The second conclusion that we point out is that to understand the representation in terms of semiotic function, as a relation between an expression and a content established by "someone", has the advantage of not segregating the object of its representation. However, since this advantage is important, we wish to point out another which is even more so. We refer to the fact that in the ontosemiotic approach, we propose that the expression and the content can be any type of object, filtered by the remaining dualities, which provides a greater analytic and explanatory capacity. Furthermore, the type of relation between expression and content can be very varied, not only the representational. For example: "is associated with", "is part of", "is the cause of/reason for" etc. The great flexibility that this way of understanding the semiotic function provides enables us not to restrict ourselves to understanding "representation" only as an object (generally linguistic) that is in place of another, which is usually the way in which representation is mainly understood in mathematics education.

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ANNEX:

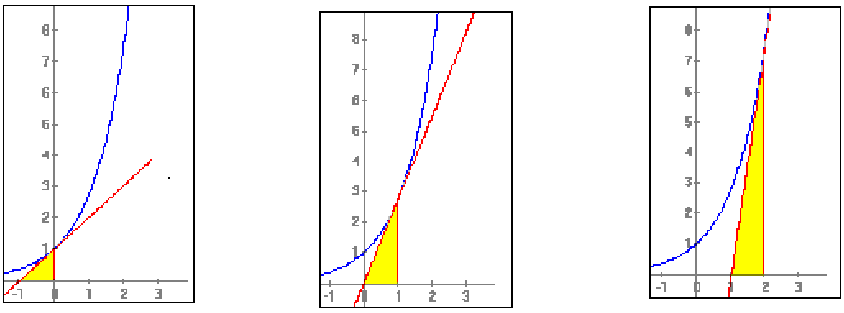
AN EPISODE OF CLASS AS A CONTEXT OF REFLECTION

Questionnaire proposed to a group of students of the first course of upper secondary school (17 years old) like part of a process of study of the derivative and the students´ four correct answers to section *c*.

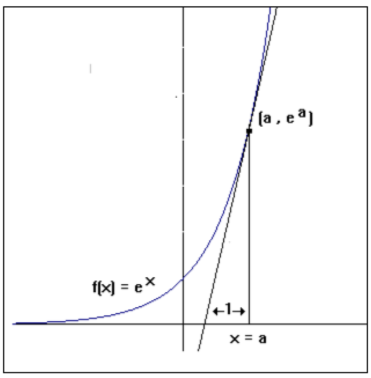
Questionnaire

In the computer classroom you have observed that the function $f(x) = e^x$ fulfils the fact that all the sub tangents are of the same length, 1. Using this property:

a) Calculate $f\,´(0)$, $f\,´(1)$ and $f\,´(2)$



b) Calculate $f\,´(a)$



c) Prove that the derivative function of $f(x) = e^x$ is the function $f\,´(x) = e^x$.

Responses to section *c*:

ALFONSO:

Slope = $\frac{e^x}{1}$

$P = e^x$

$P = f'(x)$

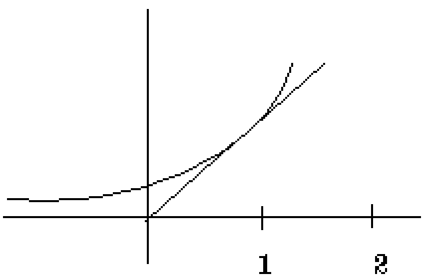
$f'(x) = e^x$

VÍCTOR

The derivative function of $f(x) = e^x$ is $f'(x) = e^x$ because the derivative of a function at one point is equal to the slope of the straight line tangent at this point.

The slope is achieved by dividing $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$, in this function $x_2 - x_1$ is always given by 1, and by dividing the vertical increment, which is the e^x, by the horizontal increment, which is 1, gives us e^x.

ALEX



All the sub tangents of the function $f(x) = e^x$ are 1, since the vertical displacement is e^x and the derivative of the function is the slope of the straight line tangent, the formula will be

$$f(x) = \frac{vertical\_increment}{horizontal\_increment} = \frac{e^x}{1} = e^x$$

ROCÍO

$$f'(x) = \frac{f(x)}{sub\tan gent}$$

es $f(x) = e^x$

$$f'(x) = \frac{e^x}{1} = e^x$$

AUTHOR CONTACT INFORMATION

Vicenç Font
Universitat de Barcelona
Facultat de Formació del Professorat
Departament de Didàctica de les Ciències Experimentals i la Matemàtica
Campus Vall d'Hebron
Passeig de la Vall d'Hebrón, 171
08035 Barcelona
E-mail: vfont@ub.edu

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