MAKING NON ROUTINE PROBLEM SOLVING A MATHEMATICS CLASSROOM ROUTINE:

A LESSON STUDY GROUP FOR BEGINNING SECONDARY SCHOOL TEACHERS[*]

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ABSTRACT

This paper presents and discusses data from a lesson study group for beginning secondary mathematics teachers. Participants in this group work in schools attended by low socio-economic status students from a wide range of ethnic, racial, and linguistic backgrounds. The lesson study group was a professional development and research initiative that engaged beginning teachers in framing, solving, and planning lessons around non-routine mathematics problems (NRMP) while exploring how these experiences impacted participant teachers' classroom practices. Included in the lesson study activities were: framing, solving, and discussing NRMP, analyzing student work, searching for NRMP in selected mathematics assessments, and placing NRMP within mandated pacing calendars and curricula. The paper begins with an overview of the lesson study context. We then discuss our framework for linking NRMP to mathematics curriculum planning and classroom instruction. Next we share data generated during three lesson study sessions. The first session concerned a sequence of non-routine problems involving paper folding; the second session focused on participant teachers reporting on their adaptation and classroom try out of paper folding problems; and the last one involved placing an assorted collection of NRMP within the secondary mathematics

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curriculum and aligning them with current standards. We conclude by reflecting on the challenges of making non routine mathematics problem solving a routine feature in beginning teachers' classrooms and discuss implications for teacher professional development.

Keywords: problem solving, non routine problems, beginning teachers, secondary school, lesson study, geometry, polygons, paper folding

1. INTRODUCTION

This paper presents and discusses data generated within a mathematics lesson study group (MLSG) attended by eight beginning secondary school mathematics teachers who work in public schools of a large urban metropolis. These are schools attended by a student population of low socio-economic status and diverse ethnic, racial, and linguistic backgrounds. As a professional development and research initiative, the lesson study group: a) engaged beginning teachers in guided joint activities of framing, solving, discussing the embedded mathematical content of, and planning lessons and lesson sequences around non-routine mathematics problems (NRMP) and, b) explored the effects of these experiences on participant teachers' classroom practices.

As a professional development initiative, this project was designed to meet the needs and respond to the realities of a group of beginning teachers working in challenging classroom contexts, as they attempted to improve their practice towards a problem-based modality of instruction. Most specifically, this initiative aimed at encouraging to incorporate NRMP in their lessons by offering them a guided opportunity to collaborate with their colleagues in planning NRMP-based lessons and units. Included in the lesson study activities were: framing, solving, and discussing NRMP, analyzing samples of student work on such problems, identifying the presence of NRMP in various assessment tools, and placing these kinds of problems within their mandated curricula. As a research study, this project aimed at exploring the effects of our NRMP-centered lesson study professional development model on participants' ability to and disposition towards making NRMP a routine feature of their teaching practice.

There is increasing concern that in urban classrooms attended by ethnic/linguistic minority and low socio-economic status students mathematics instruction focuses almost exclusively on procedural application of algorithms and formulas to solving stereotypical word problems (Boaler, 2002; Greer et al., 2009). This poor quality of mathematics instruction is seen by some as a critical contributor to social inequality (Moses & Cobb, 2001) in that it denies these groups of students access to mathematical thinking while also limiting their access to economic and other enfranchisement (National Action Committee for Minorities in Engineering, 1997; National Science Foundation, 2000). In other words, an approach to mathematics instruction that emphasizes solely routine problems is seen as unlikely to prepare students to tackle and solve novel problems in and out of school settings. While this could be blamed to a great extent to broader contextual factors such as mandated curricula and standardized testing (Haydar, 2008; Diamond, 2007; Foote et al., forthcoming), it results also from limitations in teachers' appreciation of the educative value of non-routine problem solving, their own comfort level in solving such problems, and their ability to handle the pedagogical demands this type of problem solving activity entails, particularly with regards to orchestrating whole-class discussions around multiple approaches and solution strategies in manners that engage student participation and support reflective mathematical thinking (Silver et al., 2005; Shreyar et al., 2009).

2. THEORETICAL FRAMEWORK

Our work is informed by theoretical-methodological stances deriving from mathematical problem solving (Polya, 1945), realistic mathematics education (Freudenthal, 1991), and lesson study as a model for the professional development of mathematics teachers (Lewis, 2002a; Kazemi, & Hubbard, 2008).

Influenced by the seminal work of Polya (1945), Schoenfeld (1985) and the NCTM (1989, 2000), a central aim of mathematics instruction is to develop in students the ability to use multiple mathematical tools and strategies, in flexible and reflective ways, for solving challenging, worthwhile, and open-ended problems (de Corte, Greer, & Verschaffel, 1996; Gravemeijer, 1994; NCTM, 2000; Van Dooren, Verschaffel, and Onghena, 2002). The paramount vehicle for achieving this aim has been described in the literature as: high level tasks (Stein et al., 1996), realistic modeling problem (Verschaffel & de Corte, 1997), model-eliciting tasks (Lesh & Harel, 2003), and multiple-solution connecting tasks (Leikin & Levav-Waynberg, 2007). In our research, we use 'non-routine' mathematics problem (NRMP) as a broad category that includes all of the above types of problems. By non-routine problem we mean

"A situation which carries with it certain open questions that challenge somebody intellectually who is not in immediate possession of direct methods, procedures, algorithms, etc. sufficient to answer the questions" (Blum & Niss, 1991, p. 37). Thus, the situation requires that the problem-solver adapt, combine, or invent new strategies for finding a solution (Schoenfeld, 1994, 2007).

A feature common to all of the above types of problems is their potential to engage students in reasoning, modeling, symbolizing, generalizing, schematizing, algebraizing, proving, and so on, all of which are central to mathematizing (Freudenthal, 1991).

Mathematics teacher educators have argued for organizing professional development around records of practice (Ball & Cohen, 1999). Many advocate the use of lesson study as a model of teacher-initiated and mentor-facilitated professional development that allows for collective analysis of practice (Stigler and Hiebert, 1999). Issues of adaptation of the lesson study model to the United States context have been the subject of many studies (Silver et al., 2005; Lewis, 2002a). Research on the effects of lesson study group participation on teachers' practices is also growing (Fernandez & Yoshida, 2004; Lewis, 2002b). The success of lesson study professional development modality and its spread throughout many US school districts is directing scholars to study issues of systemic scaling up of the model (Lewis, 2008). We agree with recent calls by mathematics professional development researchers who argue "for understanding the multidirectional influences between teachers' participation across the professional development and classroom settings" (Kazemi, & Hubbard, 2008).

The need for improvement in geometry instruction in the secondary grades is clearly documented in the literature (Ginsburg et al., 2005). Driscoll et al. (2007) emphasize that improving teachers' own understanding of geometric thinking is a prerequisite to their ability to foster their students' geometric thinking. The authors above developed a framework that consisted of four geometric habits of mind: a) reasoning with relationships, b) generalizing geometric ideas, c) investigating invariants, and d) balancing exploration and reflection. These habits of mind were the basis for their work to foster geometric thinking with teachers and students in grades 5-10. Along the same lines, our lesson study activities centered on geometry NRMP (cf. as an example, a sequence of paper folding problems Annex) targeting those same habits of mind. For the development of this and other units, we relied upon the Van Hiele model of geometric thinking (Fuys et al., 1988).

The diagram below (Fig. 1) describes how we frame our professional development work. The focus on teachers as problem solvers (cf. rectangular boxes) is linked to the focus on them as lesson planners and curriculum designers (cf. circular boxes). The web connecting the different boxes describes the road map that guides the design of our activities to help participants work on appropriating teaching problem-based lessons and appreciating and solving NRMP. It is important to note that the framework was revised multiple of times based on the feedback and evaluation of the participants before reaching the current format (Haydar & Zolkower, 2009). We illustrate, toward the end of this paper, an excerpt from one of the lesson study sessions that shows this dynamic aspect of the "co-evolution" (Kazemi & Hubbard, 2008) of participation between teachers' classroom practice and the lesson study which led us in this particular example to refine the framework to include activities pertaining to the "problem based curriculum" (cf. bottom circular box). The framework considers the larger picture of the middle and high school curricula. The starting point is a rich non-routine problem as defined above, (say Pn), then follow two paths that deal with connecting this problem to other problems either horizontally (connections from different mathematical strands; Pr, Ps, Pn, Pv) or vertically (sequencing the problems in order of prerequisites and difficulty; Pn-1, Pn, Pn+1). A non-routine problem becomes a source or resource for planning a problem-based lesson and, in the same way, the horizontal connecting and vertical sequencing of problems form the backbones for problem-based units. Behind the curricular side of the framework lies a strong conviction that views in an ideal good teacher necessarily a curriculum developer.

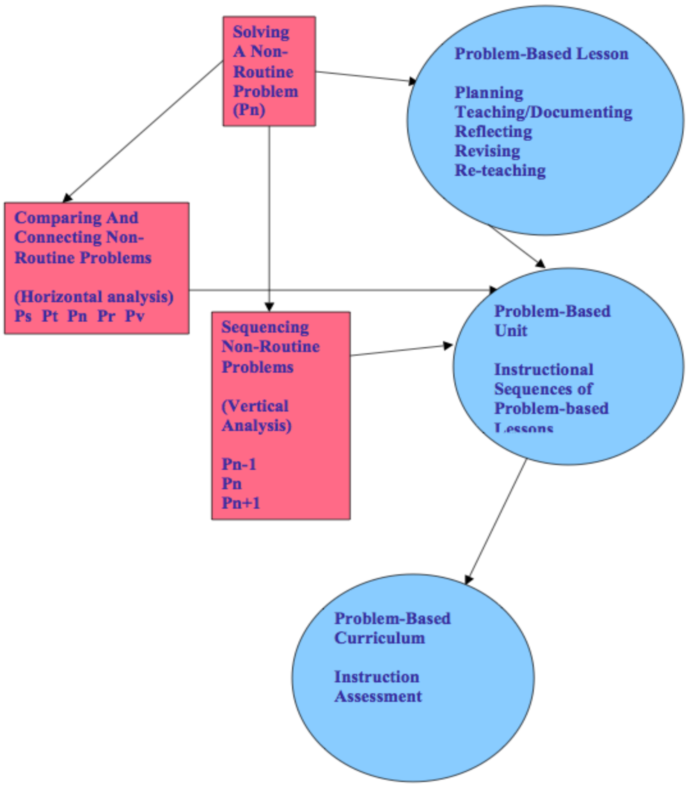

Fig.1 NRMP-Based Lesson Study Framework

The use of the above framework to select problems and relate them to other problems vertically and horizontally is illustrated elsewhere (Haydar & Zolkower, 2009). The following diagram (Fig.2) shows how an example (Pn), “the tilted square” problem, connects vertically with subsequent geometry problems (Pn+1; Pn+2; (Pn+1)’; (Pn+2)’) and horizontally with a number theory problem (Pr).

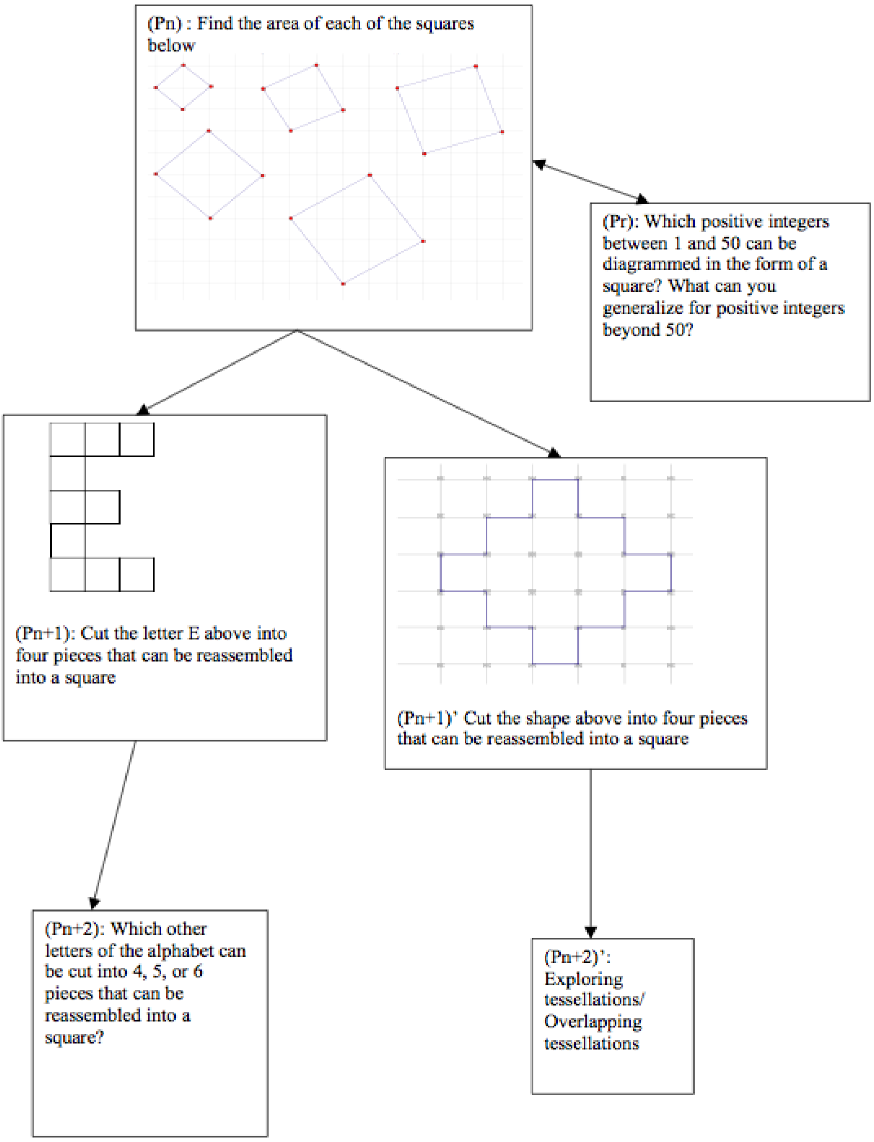

Fig.2 Example of NRMP sequence

3. METHODOLOGY

Participants in our lesson study group were eight recent graduates or in their final year in a middle school mathematics masters program in which we work. At the time when they were recruited to participate in the group, these teachers were within their first five years of teaching in what are usually referred to as 'hard-to-staff," "high needs, or "high poverty" urban schools. The majority of the students attending these schools are of low socio-economic status and diverse ethnic, racial, and linguistic backgrounds.

The study group met during nine four-hour sessions during summer and fall 2008. In line with the typical to the Lesson Study modality of professional development in mathematics (Lewis, 2002b; Stigler & Hiebert, 1999; Fernandez & Yoshida, 2004), a central focus of the sessions was designing, trying out, documenting, revising, and writing up lessons and sequences of lessons. Yet, as explained in the framework above, this central component relates to other problem solving and curriculum design focuses that expanded the list of tasks participants were engaged in. More specifically, lesson study activities included the following: solving and studying NRMP, selecting and sequencing from a list of NRMP, Designing NRMP-based lessons, trying out, documenting, discussing NRMP-based lessons, analyzing curricula in search for NRMP, analyzing assessments in search of NRMP, inventing, finding, adapting NRMP, transforming a routine problem into a NRMP, and analyzing student work samples on NRMP. The face to face meetings were complemented by online component which consisted of discussion boards and teacher narratives.

As for the structuring of lesson study sessions, for the most part this was done as follows: a) NRMP were introduced interactively with time for participants to ask questions and the sessions included b) we had extended whole-group conversations which often time occurred while participants were still in the midst of figuring out the problems—this timing of whole-group interaction allowed for ideas to be shared *in statu nascendi* thereby serving as an interpersonal gateway for thinking aloud together (Zolkower & Shreyar, 2007; Shreyar et al., 2009); c) there was an explicit focus on diagramming, both publicly on graph paper and chart paper as well as privately in teachers' own notebooks and on scrap paper; d) lesson planning activities started by brainstorming sessions according to grade levels and based on selected non routine problems; in particular, vertical curricular analysis helped participant teachers consider how the geometric topic at hand develop across grades; trying out, reporting on and revising research lessons occurred either by grade level groups or within the whole group.

All lesson study sessions were audio and video-taped. Portions of these sessions are currently being analyzed in search for evidence of an increase in participant teachers' ability to: appreciate the value of NRMP; solve and discuss alternative solution strategies for NRMP; recognize the presence (or lack) of NRMP in textbooks and assessments; design NRMP-centered lessons, and organize NRMP into a unit; and incorporate non-routine problem solving into their classroom practice.

In order to assess the effect of participating in the MLSG on each individual teacher, we collected data regarding: a) solving and explaining in detailed write ups their solution to NRMP, b) selecting NRMP from a given list of 'scrambled' problems and sequencing those problems into a unit, c) transforming a routine problem into a non-routine problem, and d) designing a NRMP-centered lesson for students in one of their classes for a given unit/topic.

We also conducted a follow up, open-ended survey four months after the last Lesson Study session. Survey questions focused on their recollection of the most memorable moments of the lesson study, narrative of what NRMP-related elements they incorporated or planning, to incorporate in their mathematics lessons, and interest in engaging, in the future similar professional development activities.

In analyzing the above data, we look for indicators of improved skills in studying, solving, and describing the solutions to NRMP as well as evidence of an enhanced ability on the teacher participants' part to search for, design, adapt, and sequence NRMP. To that end, we developed a coding scheme based on the PISA cross-disciplinary problem-solving framework and Competency Clusters (OECD, 2003). Furthermore, in order to analyze the manner in which participant teachers incorporate non-routine problem solving into lesson planning we developed a lesson template based on the Japanese lesson study and its various adaptations (Fernandez & Yoshida, 2004).

4. DATA ANALYSIS AND INTERPRETATION

In the section below, we share data generated during lesson study sessions that relates to a sequence of non-routine problems involving paper folding. This is followed by examples of two teachers presenting to the group how they planned and implemented lessons based on the paper folding sequence. Last, we share data on an activity that involved placing NRMP with regards to mandated curriculum and pacing calendars and in alignment with current curriculum standards.

4.1. Constructing polygons via paper folding

In this section, we present and discuss data from one of the two MLSG sessions in which we worked on a sequence of non routine problems involving the folding of a square sheet of paper (cf. Annex). The sequence consisted of problems asking participants to make, by folding and unfolding the square, various geometric polygons with their area described as a fraction of the whole sheet of paper. We chose to focus on the paper folding unit for the following reasons: first, due to the rich mathematical-didactical content it entails; second, it matches the focus of our MLSG on geometry and its connections to other strands; third, it offers a rich context to address the central function of diagrams and diagramming in mathematics teaching and learning; ; and finally, it serves as an example of the vertical analysis in our framework (fig.1) that can easily become a resource to plan other problem-based units at the secondary level.

The value of paper folding for teaching and learning geometry has long been noted in both recreational and school mathematics literature (Barnett, 1938; Row, 1958; Olson, 1975). The mathematizing and didactizing potential we identified in the paper folding sequence was as follows: geometric transformations; dissections of a square; moving from folding to mental folding; using the properties of polygons to construct them by folding; measuring, estimating and calculating; describing, explaining, proving, convincing; moving from informal proving to formal proofs; folding and diagramming: transitioning back and forth from folding/unfolding to working with the resulting lines and shapes, to recognizing and using the underlying templates; making diagrams and using them as thinking devices; using re-allotment and other strategies in order to find the area of polygons; and, last but not least, recognizing the templates that underlie the folds and using these to obtain, via folding, the requested polygons.

4.1.1. From Describing, to Folding, to Diagramming: "An expensive phone call" (Episode #9) Problem: Make a rectangle that is 3/8 of the square sheet of paper

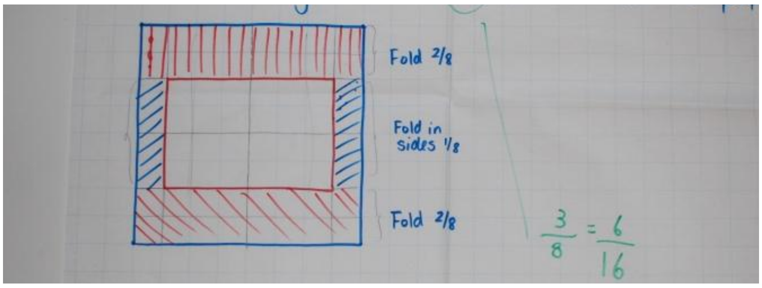

Fig. 3: Make a rectangle with an area that is 3/8 of the area of the sheet of paper

After Amy finishes presenting her solution and explaining her diagram (fig. 3)

9 F (facilitator): What's nice about this diagram is that the square and the rectangle are concentric, right? But do you think it's possible to do it in a different way, so that two of the sides of the rectangle coincide with two of the sides of the sheet of paper? Is it possible? Can you imagine it? Can you visualize it? … I don't know if my question makes sense. I'm thinking… there must be a way to do it in another way, so that these sides of the rectangle touch sides of the sheet of paper (*pointing to various parts of Amy's diagram*).

10 Sharlene: Yes, I did that

11 F: Yes, you did that one? Can you show it to us?

12 Sharlene: I just folded the paper in half and divided the eighths first and then-

13 F: Oh! I have a nice exercise. Sharlene is going to talk and someone is going to make a diagram while she's talking. Let's do it over here (*points to the blackboard*). So, we're going to make another rectangle that has an area of 3/8.

Dolores goes up to the blackboard, takes a marker, and draws a square on the chart paper sheet posted on it.

14 F: Ok. So, what are the instructions?

15 Sharlene: Divide it into two equal parts (*draws a vertical line joining the midpoints of the square's horizontal sides*)

16 Sharlene: Then divide each half into… quarters

Dolores begins drawing a horizontal dotted line

17 Sharlene: Hold on…. Stop, stop

Laughs. Dolores turns facing Sharlene who is showing her the folds on the patty paper.

18 F: You can't show it to her! You're talking over the phone

19 Sharlene: Ok… I can't show it to you. So, you have the horizontal line half… yes horizontally halfway and vertically. It's half, so these are midpoints, mid-segments. So this divide it into four equal parts … Now take the midpoint from the… first quarter… top midpoint, from any one of them, and connect with the bottom midpoint of *(looks at Dolores and shake her head)…* Yes, straight up. Now-

20 F: But you're talking on the phone!

21 Sharlene: To that midpoint and to the midpoint of the other top quarter to the bottom. And this divides it into eighths (*gesture…)*

22 F: Wait, wait, wait. You're talking on the phone. They're charging you by the minute. It's a long distance phone call. And it's getting super expensive. (*Laughs*) Could you have told Dolores to do the same but in just one sentence?

23 Sharlene: Fold in half then fold over again then… fold

24 Leila: Let me try it. Fold the paper into four…No… Fold the paper until you get a four by four

25 F: Mm… A four by four… Four by four like when you're driving a pick up? (*Laughs*)

26 Leila: I mean… Four squares-

27 F: Can we say divide the paper into a 4 by 4 array of smaller squares?

28 Leila: Yes.

Dolores draws the square divided into only eighths again and then shades three rectangles from the top right (fig. 4).

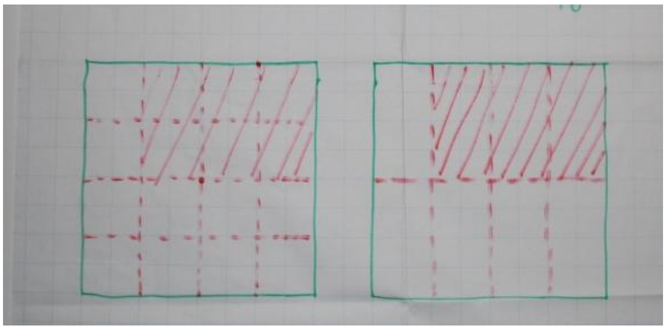

Fig. 4 Dividing the square into fourths and eighths

The fragment above concerns a participant's attempt at describing how she obtained a rectangle of area 3/8 from the square sheet of paper. In order to create a situation that would force Sharlene to produce accurate, step by step instructions for Dolores to make a diagram of her folding without showing her folding actions, the facilitator proposes, as an imaginable scenario, a long distance phone conversation. Sharlene now tries to abstain from using ostensive language and gesturing. She continues giving instructions for Dolores to draw the folding lines on the square drawn on the chart paper posted on the blackboard. Yet her instructions are rather cumbersome. In order to push for a more abbreviated description of the folding process, one which could be translated easily into diagrammatic form, F brings in, as an imaginary constraint, the need for the phone conversation to be very short. Can the instruction be given in just one sentence?

As we see it, this exchange is paradigmatic of a tension between folding as an embodied, temporal action (first fold it like this, then like that) and the folding lines that result from those folding actions. Interestingly, folding a square sheet of paper into 16 congruent squares involves 6 straight lines (3 horizontal and 3 vertical), yet the process of doing this via folding only requires 4 folds. Enter Leila. While Leila's version is, indeed, a more abbreviated one and thus meets the expensive phone conversation constraints, yet her use of language is a bit vague ("until you get a four by four"). F replaces 'fold' with 'divide' and adds the word 'array' thereby suggesting: "divide the paper into a four by four array of smaller squares."

4.1.2. Between Folding and Diagramming: Dolores and the Regular Octagon (Episode #10)

Problem: Fold a square sheet of paper into a regular octagon. What is the area of this octagon as a fraction of the area of the square sheet of paper?

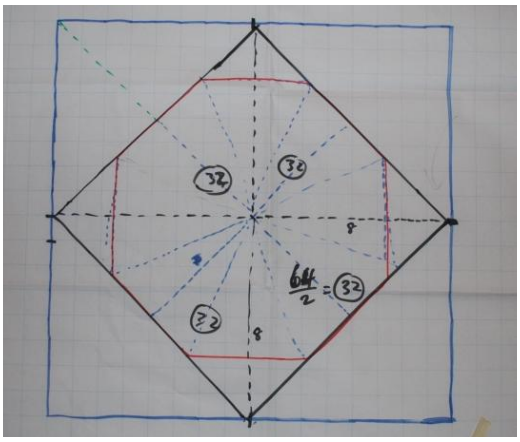

Fig.5: Folding the square into a regular octagon

[NOTE: The dotted lines were added by F later on, during the course of the exchange around Dolores's presentation of the folding process that led her to the regular octagon]

1 Dolores: This is supposed to be a regular octagon. It is here. (*Goes back to her seat and picks up the square paper on which she has done the folding).* But I couldn't transfer it back on the diagram.

2 F: You're claiming that this is a regular octagon. Do people buy this idea? Is it true? (*Everyone nods in agreement*). So this means that this side here (*points to the top horizontal side and the one adjacent to the right*)-

3 Dolores: Actually when I folded it I measured it

4 F: So you did it approximately… you measured it. Was that allowed?

5 Michael: (*Comes up to the board and points at the octagon's vertices)* How did you get these points? Can you explain how you got them by folding?

6 F: Yes, how did you get them in a way that you know for sure that this is a regular octagon? How do you know for sure that the distance from here to here is the same as this side (*pointing to two consecutive sides of the octagon in Dolores's diagram)?*

7 Dolores: I am not sure about it here *(points to the diagram)* but I am sure about it here *(shows the folded paper)*

8 F: So, why don't you give us the instructions and we'll try to fold it the way you did it?

9 Carol: I am thinking maybe because the diagonals are the hypotenuses of right triangles. (*Comes up to the board and points to various parts of Dolores's diagram*). Is this fold half of that one? …. Never mind. Forget what I said… I thought that fold was exactly two units

10 F: (*Addressing D*olores) Maybe you can do it step by step with somebody folding the same time as you did it.

11 Dolores: I imagine this is a circle (*points to her diagram*) and the apothem is like the radius. And I took this paper (*shows her folded paper*) and folded it in half and that's how I got this line here (*points to the side of the octagon on the diagram*)

12 Chorus: <u>What?!/How?</u>

13 Dolores: It's so hard to explain!

She then takes the paper folding square and shows step by step how she did the folding with everyone attempting to follow the same folding. After about 10 minutes, Dolores comes back and takes a circular plate from the snacks table and traces the following diagram (fig. 6)

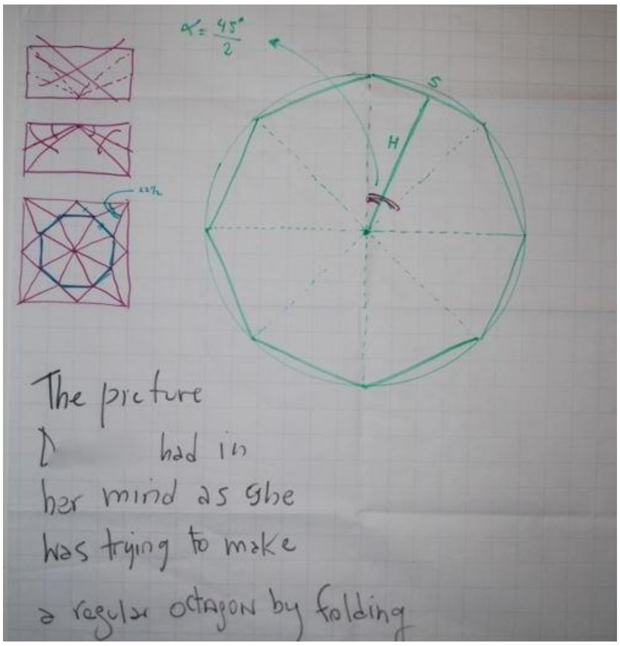

Fig. 6: Relating the regular octagon to the circle

F completes the diagram by drawing the 22.5 degree angle and writes: ***"The picture that Dolores had in mind as she was trying to make a regular octagon by folding"***

The diagram Amy presents to the group does not include all the folds she made. F makes a point about that by adding those 'invisible lines.' This exchange is paradigmatic of a tension between: the diagram as an inscription of the final product (a rectangle concentric to the square) and the diagram as inscribing the step by step procedure that led to that product. We have identified this as a blind spot in the math-didactical skills of (beginning) mathematics teachers. In light of this, we view geometry activities such as this sequence of paper folding problems as productive opportunities for strengthening such pivotal skills.

Engaging teachers in folding, mental folding, and diagramming fosters their relational understanding of geometric figures and their properties. The paper folding sequence creates opportunities for them to revisit those relationships which are "processes of being and having" as "processes of doing" (Halliday, 1994; Zolkower & de Freitas, 2010). One example from the lesson study sessions that illustrates this point is when teachers worked on problem 6.4 (cf. Annex) *[6.4. Folding a square sheet of paper into the largest possible equilateral triangle]* Although they all knew and taught the definition (process of being and having) of equilateral triangle multiple of times, as one in which all sides have the same length and also as a regular polygon with all angles measuring 60° they had a real challenge in folding and making the largest possible equilateral triangle (processes of doing). These as well as other similar paper folding activities help beginning teachers develop flexibility in moving back and forth among relational (mathematical) and material meaning-making.

4.2. Reporting on paper folding classroom activities

As described throughout the paper, our professional development was designed to center around teachers' own practice with the aim of inspiring them to and supporting them in using NRMP in their lessons. We were therefore interested in exploring the links between "the knowledge and ways of knowing that teachers develop[ed] as they work [ed] with colleagues in PD with what happens as teachers tr[ied] to enact their learning in the context of their classroom teaching" (Kazemi, & Hubbard, 2008; p. 429).

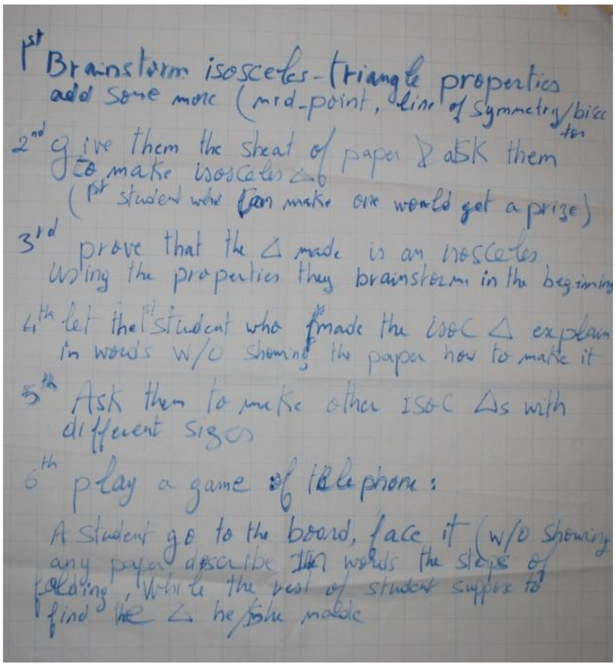
1st Brainstorm isosceles-triangle properties
add some more (mid-point, line of symmetry/bisector
2nd give them the sheat of paper & ask them to make isosceles △
(1st student who can make one would get a prize)
3rd prove that the △ made is an isosceles using the properties they brainstorm in the begining
4th let the 1st student who made the isoc △ explain in words w/o showing the paper how to make it
5th Ask them to make other isoc △s with different sizes
6th play a game of telephone:
A student go to the board, face it (w/o showing any paper) describe in words the steps of folding, while the rest of student suppose to find the △ he/she made

Fig 7: Initial brainstorming during common planning session

Following are two examples of teachers reporting on how they adapted aspects of the paper folding sequence in their 7^{th} and 6^{th} grades:

4.2.1 A Paper Folding Lesson in a Grade 7 class (Leila's Narrative)

Below we share an extended monologue whereby a teacher participant (Leila) describes to the group a paper folding lesson she had recently done in her 7^{th} grade classroom. The lesson was a part of a unit that Leila and two other participants collaborated on planning based on the paper folding sequence we had worked on during two prior sessions (Fig.7). They selected and adapted a sequence of activities to address Leila's students' needs and meet the 7^{th} grade standards that she was working on at the time. As in any lesson debrief, the group discussed Leila's lesson and came up with suggestions for improving the lesson.

On the table, in front of her, Leila has placed a stack of student work samples and, in her hand, she holds a narrative of her lesson which includes also her lesson plan. Below is the transcription of her extended monologue:

> The lesson was about finding geometry in paper folding. It was right after teaching areas of polygons. The lesson took about 50 minutes. […] I showed the students this paper (*shows a square sheet of paper*) and said "Find a square whose area is one fourth of this square." One student said: "But we don't know how big the square is!" Another one said "We need to measure it." I replied "Don't measure it. I'm not asking for the exact measure. But you need to find a square which is one fourth of the big square. If we add four squares of the type you find we end up with this big square."
>
> Even though I gave the instructions many times, one student started measuring the paper. He took a ruler and began measuring. Other students were looking at the paper then looking at each other like as if I was saying something crazy (*smiles*). And some others started doing the actual folding. One student folded the paper in half (*folds the square in half resulting with a rectangle*) and showed it to me. I used this as a hint. I said "You're on the right track" and I asked "What's that

shape?" The student looked at it and said: "A rectangle." I then asked "This rectangle is how much of the big square?" He was like "Half." I said "Ok, you're close. This is half. Can you find one fourth?" He asked "Can I do this?" (*Folds the rectangle in half*). I said "Yes." He folds the paper and looks at it. Then I asked him to open it. I colored one of the small squares in red and asked "How much is this square? The area of this square out of everything (*points to the whole paper*). They said "One fourth." … I think this was my mistake… as soon as this student folded it twice I said "This is correct."

And then everybody started doing the same. We spent about 10 minutes on this part. Then I asked them to find a square whose area is 1/16 of the original square and that was easy. The hardest thing was to find the 1/4. Then we did 1/16 and they found it and we kept folding until we got 1/64 and this is their work (*shows a square with traces indicating the folding into 64 equal squares*). After that, I drew on the board how we got from the square to the rectangle and I asked them to draw what they did in folding and that's what they did. (*Hands out samples of student work for the group to look at and gives them five minutes to look at them*).

When I asked them to describe in words what they had done, one student said "Fold it like this" and I said like what? (*Turns left*). Pretend I am not seeing you. I am in another room or over the phone. The student comes to the front puts the rectangle close to my eyes, "Like this," he says. I say, "I cannot see. Use other words. Imagine I'm in another room and you are describing it to me over the phone." The student said "Fold it to make a rectangle" I said "Ah, ok… I did it." She said "Now fold it top to bottom." I did this (*folds the rectangle over the shorter sides to get other rectangles*). She said "No, that's not what I meant. Fold it right to left." I said "Ok" and turned it like this (*turns the rectangle 90 degrees to have the short sides meet horizontally*). She says "No, fold it into two" and I said (*pointing to her rectangle*) "This is into two." She said "No, not like this; fold it into two." I said "Use the dimensions to describe your folding." "She was like "What do you mean by dimensions? What are the dimensions?" I said "When you have rectangle, how many sides you have? What do you call this one and the other?" Another student said "Length by width" (*laughs*). I said "Ok, use the words length and width." She said "Fold the rectangle by length" and only then I folded it like this (*getting the ¼ square*). This was the first part of describing their folding. As soon as they got the first part it got easier to do the other parts.

So, this is how one student described her folding (*reads from a student's write up*): *"The length and the width equal s. When the square is divided by ½, the length will be s. But the width will be equal s/2. When you fold the square in half along the length the length will be s/2 and the width will be s/2 when the length and the width you get* $(s/2)^2$. *When you simplify the equation, you will get* $s^2/4$, *which is the area of the big square divided by 4. Now that the length is s/2 and the width is s/2, divide the square in half the length will equal s/4 and the width will still be equal s/2. When you fold it along side the length in half the length and side will both be equal to s/4. Then s/4 times s/4 is* $(s/4)^2$ *So when you solve the equation, your answer should be* $s^2/16$.*"*

In her presentation Leila relied on her students' work, her own narrative, and her lesson plan to reconstruct to the other participants what happened in her classroom. This was how the group attempted to overcome the disadvantage of not being able to visit each other's classrooms to observe the commonly planned lessons. We note in the Leila's narrative some mirroring to what happened while the teachers were solving the same task (see above). This was found in many debriefing instances. As a result of solving and studying NRMP, teachers expressed a better understanding of what their students go through when they solve mathematical problem.

Leila's lesson showed a connected grasp of mathematical content where a geometry problem was used in the context of a fractions and algebra lesson. In her online reflection, Leila wrote *" I chose this question because I was teaching fractions and because the 7th graders are having trouble understanding expressions such as*

'1/4 of x.'…The paper folding activity helped them have a better understanding of variables as well as looking at fractions in a different way.'

4.2.2. A Paper Folding Lesson in a Grade 6 Class (D's Narrative)

Below is an online reflection that Dolores wrote about her paper folding lesson with her 6th grade inclusion class.

…From a previous lesson on *Archimedes Stomachion Puzzle* (Note: This was another geometry problem we worked on in our lesson study group). I discovered that most of my students had prior knowledge about identifying and describing certain polygons. I taught them the characteristics of a few polygons before the test – by mainly drawings and listing the characteristics. There was no time for them to manipulate and discover features for themselves. Many 6th grade students have difficulty drawing straight lines even if they have rulers. They have trouble measuring also. I notice when they have to rule up their journal books, they cannot draw parallel lines for the columns. Sometimes, they cannot draw a polygon (or a 2-D shape) even if the dimensions are given. I have noticed them doing some drawings for technology class, but they used specific types of paper (e.g. grid paper). Paper folding is a good opportunity for students to build confidence in measurements and describe shapes.

During the lesson, I walked around the class and observed what the students were doing. Some of them seemed unsure of what they were doing. One student called me to verify the characteristics of the shape she wanted to fold. A few students observed what others were doing before they started theirs. Everyone seemed motivated to fold. One student in particular worked quickly. He folded a kite. I asked him how he knew that the sides of that kite were congruent. He quickly folded along a line of symmetry and said "Yes, they are." I asked him for the name of the line he just created and to compare the measurements of the sides and angles. He replied "A line of symmetry." At that point, I showed the kite to the class, and I asked how they could verify that the shape is really a kite. They responded correctly. I reminded them that they had to write an explanation about what they had done. I continued to walk around the class to observe them at work. Most of the students paid attention to the measurement of the sides and not the angles.

A few students mentioned asked for help. Student D. moved over and offered assistance in making a kite. Following this, some students teamed up to create shapes. I said to the class that I wanted everyone to be able to eventually fold a regular octagon. Many students did not seem to follow any sort of mathematical reasoning. All students who attempted to make a regular polygon by paper folding used a guess and check (or trial and error) method.

Nevertheless, I was most satisfied with the high level of engagement. Everyone was actively engaged in the paper folding activities. I have two students that are usually non-compliant during class. However one of them emerged as a leader in helping others. The students were not able to describe how they did the paper folding. I promised that we would continue the paper folding another time. […]

D, who has been identified as mildly autistic and usually engages in compulsive behavior, says that origami is one of his hobbies. Although he has been classified also as emotionally disturbed, I noticed that when he does origami, he seems at peace. The class had a group project that consisted of creating math games. D used paper folding to create little paper containers for all the groups. He made all his game pieces by first folding and then colored them differently. Furthermore, he boasts of being able to make 16 different models of planes by paper folding […]. From September on, his grades have gone up. I am expecting him to score high on the math test. I have taught inclusion classes for the past few years, and have noticed that special education students perform very well in hands-on geometry activities.

Examining how different teacher participants appropriated the paper folding sequence and how the lesson collectively planned during a study group session unfolded in different classrooms is in line with recent professional development methodological recommendations calling on researchers “to identify a practice that is the focus of the PD effort, track how teachers reason and work with that practice as it travels to their classroom, and track how they reason with that practice when they return to PD. The key methodological concern is connecting adaptations back and forth over time” (Kazemi, & Hubbard, 2008; p.435).

4.3. Situating NRMP in Relation to the Curriculum Standards

The curricular dimension of the framework discussed in this paper is intended to support teacher participants in locating a given NRMP (Pn) with another that should precede (Pn-1) and/or follow after (Pn+1), as mentioned above this vertical analysis constitutes the backbone for designing problem-based units of instruction. Also, NRMP are analyzed and compared horizontally to problems from other mathematical strands.

We found that most teacher participants come to value the NRMP-based lessons and gain skills in incorporating some aspects of them in their lessons with varied level of success and sophistication among individual participants. Most of them do voice however concerns about the time that it takes to teach such lessons and a lack of self confidence in being able to prepare their students to the State testing using this NRMP. We present here an example on how one MLSG activity opened a channel for teachers to voice their concerns about teaching NRMP and how this discussion generated future broad curricular activities to map the NRMP with the New York State standards and led us to revise the framework to include this curriculum dimension.

In one of the debriefing sessions participants were sharing some of what they were doing to incorporate NRMP in their classrooms. One teacher, Michael, while talking about an NRMP-based activity, shifted the conversation toward the issues he was having with this manner of teaching mathematics:

“*I guess the issue I am getting at is that we tend to… or we are trained to teach things in very orderly way today we are going study radius, tomorrow functions the next day probability ... everything is treated in very discreet way… If you take* *these, something like a very rich problem it's all over the board ..ok we're hitting these nine standards and then you hit the issue of some kids might look at this and do it this way and some might do it this way, solve it algebraically or solve it geometrically. Then you might take a problem and you might say: I am going to hit all these nine standards. But the class may only hit one of them. I think…I don't think it's impossible and it's worth the attempt. But it makes it difficult in the mind of a teacher to come up with something that will create a curriculum and help the kids to pass the test. It's almost like asking people to take a leap of faith and when there are all these other pressures.. How do I do that? We're attempting to take all these rich problems and fit them to our rigid… sort of like our idea of how you need to teach this first then this then this there is intrinsic difficulty in that?"*

The other teachers in the group responded to Michael's voicing of the problem with what Little and Horn (2007) found as fairly common among teachers in similar groups which is trying to "normalize" problems of practice by marking them as an expected aspect of teachers' work" (Little & Horn, 2007; p.82) Like in many other instances during our MLSG discussions, normalizing the problem of practice of teaching NRMP within this group did not turn the conversation away but "toward teaching as an object of collective reflection" (Little & Horn, 2007; p.82). Individual teachers shared how they dealt with paradigms tension but also with the testing and standards' demands that might hinder their teaching and learning of NRMP. The result of this collective reflection led to a discussion on "hybrid pedagogies" (Bernstein,1990), namely instructional approaches that engage students in framing and solving non-routine—hence, weakly classified—mathematics problems yet supports them via waves of strong and weak framing (Zolkower & de Freitas, 2010).

In order to help our teachers look at the larger curricular picture and situate the NRMP in relation to the state and national standards and testing, we designed activities that fit under the problem-based curriculum part of our framework (Fig.1). Participants were engaged in analyzing curricula and comparing international, national and state assessments in light of NRMP. In particular, one of these activities involved situating a purposely 'scrambled' collection of NRMP, including some of those the group had worked on in prior lesson study sessions, in relation to the grade level State standards. The state standards were posted on chart paper all around the room and participants posted the problem cards prepared for this activity where they saw a matching. This helped highlight how one NRMP covers necessarily more than couple of standards and often connects standards from different mathematical strands. We look at this as an example of a “dialogue [that] does more than simply report on or point to problems of practice, but supplies specific means for identifying, elaborating and re-conceptualizing the problems that teachers encounter and for exposing or generating principles of practice” (Little & Horn, 2007, p. 81).

5. CONCLUSION

This study suffered from two limitations. First, we did not have direct access to participants’ classrooms. Thus, what we were able to gather about participants’ teaching was obtained via self-reporting and accompanying classroom-generated artifacts. Second, the length of the study was relatively short during which we had face-to-face, 3-hour long bi-weekly sessions complemented by asynchronous on-line interaction. Notwithstanding these two limitations, we believe our exploratory study yielded interesting findings that contribute to the central problematic of supporting teachers’ efforts at changing their practice towards a problem-based modality of curriculum and instruction.

In line with other studies on professional development that engage teachers in learning and teaching non-routine problem, we found that NRMP-based MLSG is a safe professional development modality strengthening teachers’ own mathematical-didactical knowledge as well as their problem solving. Incorporating NRMP on a routine basis into mathematics lessons poses high demands on beginning teachers as NRMP require a connected understanding of mathematics where isomorphic connections between different mathematical strands and structures are developed and used to solve the problem. As one participant put it in her end-of-the-project written reflection, *“The engagement in solving non routine problems so resembled what happens in my classroom. Students demonstrate different approaches as did we as teachers. The environment needs to be safe so that all feel comfortable to share. I also rediscovered my own strengths and weaknesses as a problem solver.”*

We found uneven results in how individual participants incorporated NRMP in their lessons this could be linked to participants' disposition to try these lessons in their classroom and the support they have in schools. These differences echo the three typical trajectories for the co-evolution of teachers' participation across the professional development and classroom settings as documented by Franke and Kazemi (2001):

- participating in the workgroup was a separate and distinct activity from their classroom practice.
- The workgroup was a place to gather ideas for the classroom, but it did not represent a setting to bring and work on instructional dilemmas
- The workgroup setting was a place to puzzle through complicated or confusing student strategies and to discuss instructional questions and observations.

The NRMP-Based Lesson Study Framework and the designed activities helped participants shift from thinking about incorporating non-routine problem as potential tasks that beginning teacher need to do after they become expert teachers to thinking that this is essential part of being a better mathematics teacher. In terms of Bernstein's (1990) theory of pedagogic discourse, NRMP are weakly classified tasks since by definition "non-routine problems do not have a straightforward solution, but require creative thinking and the application of a certain heuristic strategy to understand the problem situation and find a way to solve the problem" (Elia et al. 2009, pp 606-607). Incorporating these kinds of problems on a routine basis into mathematics lessons poses high demands on beginning teachers. This includes having a robust and inter-connected knowledge of mathematics (Ma, 1999) and being able to enact hybrid pedagogical practice which guide students' mathematizing with waves of strong and weak framing throughout different phases of the inquiry processes. Therein resides the necessity of creating guided opportunities for beginning teachers to collaborate in working on NRMP, discussing their instructional worth, identifying the content embedded therein, and figuring out when, where, and how to place these kinds of problems into the curriculum they and their students routinely enact in their classrooms.

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Annex: Finding Geometry in Paper Folding

0. Measuring without a Ruler and Making a Square

0.1. Without using a ruler, find exactly 6 inches on a sheet of 8 ½ by 11 paper. Is there more than one way to do it? Use diagrams to show each of the ways you found.

0.2. Now that you have located 6 inches on your sheet of paper, use folding and cutting to create the largest possible square out of it. No ruler is allowed! How are you sure that your square is actually a square?

1. Making Squares

1.1. Fold a square sheet of paper (e.g. patty paper) into a smaller square such that its area is ¼ of the original square. Record the folding process step by step. How do you know you have constructed a square? How can you be certain that the square you constructed has an area that is a 1/4 of the original square?

1.2. Write down a convincing explanation that the area of the smaller square is ¼ of the area of the larger square.

1.3. Pair up with someone in the class to share each other's folding and writing work. Are you convinced that your partner has constructed a square with an area exactly ¼ of the original square? Why or why not? If not, what modifications could be made on the written explanations and accompanying diagrams?

1.4. Working with your partner, construct an explanation of the folding process done in 1) and its outcome. You are not allowed to use words; only diagrams, numbers, arrows, and letters.

1.5. Fold a new sheet of square paper into a square whose area is 5/8 of the total area of the sheet of paper.

2. Making Non-Square Rhombi

2.1. Fold a new square sheet of paper into a non-square rhombus so that its area is ¼ of the original square. Record the folding process step by step. Write down a convincing explanation: a) that the shape is indeed a rhombus and b) that its area is ¼ of the area of the larger square.

2.2. Pair up with someone in the class to share each other's folding and writing work. Are you convinced that your partner has constructed a non-square rhombus and that its area is exactly ¼ of the original square? Why or why not? If not, what modifications do you suggest in the written explanations and accompanying diagrams?

2.3. Working with your partner, construct an explanation of the folding process done in 2.1. You are not allowed to use words; only diagrams, numbers, arrows, and letters.

2.4. Fold a new sheet of square paper into a non-square rhombus whose area is 1/8 of the larger square. Explain how you know this is the case.

2.5. Fold a new sheet of square paper into a non-square rhombus whose area is 1/2 of the larger sheet of paper.

2.6. Starting with a new square paper, fold it into a non-square rhombus such that it has a larger area than the one in 2.1. What fraction of the sheet of paper is this new rhombus?

3. Making 'Kites'

3.1. Fold a sheet of paper into a 'kite' such that has the same area as the rhombus you made in 2.1, that is, ¼ of the area of the sheet of paper. Explain in writing how you are sure that both shapes have the same area.

3.2. Using the same sheet of paper, fold it in such a way that you make a 'kite' with an area that is smaller than the rhombus. Convince yourself and others that this is indeed the case.

3.3. Make a 'kite' with an area of ½ of the sheet of paper

4. Making Octagons

4.1. Take a new sheet of square paper and fold it in such a way that you make a regular octagon. What is the area of this octagon expressed as a fraction of the area of the square sheet of paper?

4.2. Can you make a new octagon with a larger area? Can you make one with a smaller area?

4.3. Take a new square sheet of paper and fold it into a square whose are is ½ of the larger square. Now using the same folds, add four more identical folds so as to obtain an irregular octagon. What fraction of the area of the larger square is this octagon?

5. Making Rectangles

5.1. Take a new sheet of square paper and fold it into a rectangle. Find the area of the rectangle as a fraction of the total area.

5.2. Using the same sheet, fold it into a rectangle such that its area is 3/8 of the paper.

5.3. Make a rectangle by folding a new sheet of paper such that it is titled on a 45 degree angle with respect to the sheet of paper. Show that the area of this titled rectangle is the same as the area of the one you made in 5.2.

6. Making Triangles

6.1. Make a right triangle with a square sheet of paper such that its area is 1/64 of the sheet of paper. How many folds did you make? How many folding lines resulted? What can you generalize from that?

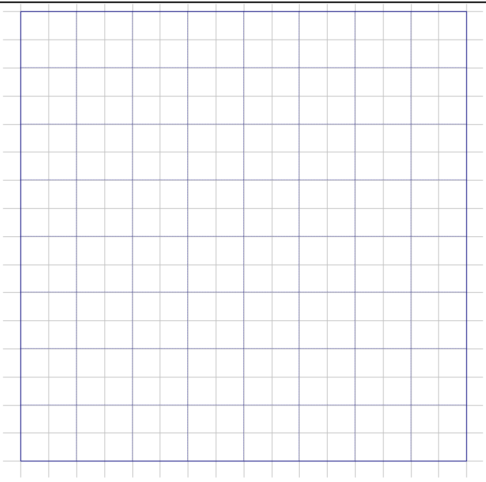
6.2. Make an isosceles triangle such that its base coincides with one of the edges of the square sheet of paper.

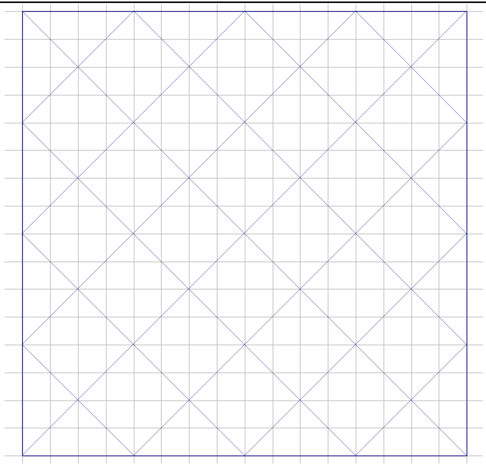
6.3. Make an equilateral triangle and find its area expressed as a fraction of the area of the sheet of paper. How many folds did you use for making it? Can it be done with just two folds? Try it!

6.4. Find the largest possible equilateral triangle that may be made out of a square sheet of paper of given dimensions. Describe a method for making any equilateral smaller than that one on the same sheet of paper.

7. Making Other Shapes: Try out different kinds of folds to see what other geometric shapes you can make. On the basis of the new shapes you have found, add more activities to this ‘geometry and paper folding’ instructional sequence.

8. Use the two templates below to create other paper folding activities





9. Where is the mathematics? Make a list of all the mathematics you encountered in these activities. Be as specific as you can.

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