PROMOTING STUDENT REASONING THROUGH CAREFUL TASK DESIGN: A COMPARISON OF THREE STUDIES

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ABSTRACT

Researchers have found that students as young as elementary school can engage in mathematical reasoning. Specifically, particular tasks tend to encourage this reasoning. This paper provides insight into some general characteristics of tasks that may lead to arguments that represent varied forms of reasoning. In this paper we report on arguments built by diverse student groups, of different ages, that were used to justify their solutions to problems from the fraction and counting strands of longitudinal and cross-sectional studies. We compare the characteristics of the two tasks and suggest how the implementation of tasks such as these can help elicit varied student reasoning.

Key Words: Reasoning, Justifications, Task Design

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1. INTRODUCTION

Recently, mathematicians and mathematics educators have called for an increased emphasis on reasoning and proof at all levels of the curriculum. This focus on reasoning and proof is a crucial component of elementary and middle-grade mathematics learning, in that reasoning and proof are the foundation of mathematical understanding and are necessary for acquiring and communicating mathematical knowledge and building sustainable understanding in mathematics (Hanna, 2000; Hanna & Jahnke, 1996; Polya, 1981; Stylianides, 2007). Researchers have shown that children as young as eight and nine years old make conjectures, and justify their claims with sound arguments if they are supported in the classroom and afforded opportunities to reason collaboratively (Author, 2005). Author and Other (1996) traced the development of an elementary student's mathematical reasoning while attending to counting tasks over 5-year span. In grade three Stephanie was presented with the task of finding all possible towers four cubes high when selecting from plastic cubes in two colors and this task was extended and built upon in the fourth and fifth grades. Author and Other found that Stephanie's justifications progressed from the use of trial and error and finding patterns to elaborate direct and indirect proofs. By grade five Stephanie created a detailed written version of "proof by cases" to show all possible towers n-high that could be built. Another example can be found in a year- long study in an elementary classroom, where Lampert (1990) found that students moved back and forth between inductive and deductive arguments while working on a task involving finding the last digit in multi-digit numbers without doing multiplication. In justifying their assertions, the students were able to provide proofs about the patterns and interpret each other's justifications.

Unfortunately after over ten years, we are still only beginning to understand how students' mathematical reasoning develops and which environments can best support the development of student reasoning (Yackel and Hanna, 2003). Author (2008) suggests that children as young as eight years old can develop arguments that take the form of mathematical proof; however, this act is not instinctive. Instead this argumentation emerged as teachers questioned and prompted students to justify solutions. Once students did begin to naturally justify solutions, they expanded their arguments in response to the challenge to convince themselves and others of the reasonableness of their arguments.

In this paper we attempt to identify general task characteristics that may elicit specific forms of reasoning. This is accomplished by studying in detail, using a modified grounded theory approach, the nature of student reasoning across three settings and across elementary and middle grade populations. The settings are diverse in that they range from low to high socioeconomic, and across urban, working class, and rural/suburban environments. We report on children's arguments that represent forms of reasoning across these ages and student groups that were used in justifying solutions to problems from the fraction and counting strands of longitudinal and cross-sectional studies. Analysis of children's arguments indicated that certain tasks tended to elicit particular forms of reasoning across all age groups and populations. Interest in studying the commonalities among these tasks led to the following question: Are there identifiable characteristics of mathematical tasks that might contribute to the co-construction of meaning and the development of varied forms of reasoning?

2. THEORETICAL FRAMEWORK

2.1 The Notion of Proof

Many researchers stress the role of discourse in the mathematics classroom to encourage reasoning and proof (Balacheff, 1991; Hanna, 1991; Author, 1995, 2008). Other and Author (2005) highlight the importance of emphasizing justification of solutions to problems in school mathematics, rather than formal proving. They argue that the rigor and form required for formal proof is not accessible to young children who have not yet acquired the algebraic tools to represent their arguments. They posit that by encouraging the use of informal justification, students will have the opportunity to engage in proof-like activities before writing formal proofs using symbolic notations that enable them to express their ideas in general form. They argue that it is important for students to grow accustomed to the necessity of convincing others of the validity of their ideas so that “proof-making” can become an integral part of the mathematical process of problem solving activity.

Stylianides (2007) suggests a definition of proof as it applies to elementary school mathematics. She proposes that proof in school mathematics can be identified when a sequence of assertions fulfills three conditions: It must use statements that are accepted by the classroom community as true, which may be definitions, axioms, or theorems; it employs forms or modes of argumentation that are accepted by the community as reasonable or within their conceptual reach; and it is communicated using representations or forms of expression that are understood by the community. She emphasizes that what may constitute a proof in a high school class may not be valid in an elementary school classroom, if it uses terms or forms of reasoning that are outside of the students' experience or ability. By introducing proof in this way, students who are exposed to mathematics in elementary school should get a sense of what mathematicians do. Students gain an authentic mathematical experience when they are allowed to use forms of argumentation to the best of their cognitive ability while retaining the accepted modes of argumentation and methods of proof that are used by mathematicians. Stylianides (2007) suggests that the notion of proof is dependent upon the community in which it emerges, indicating that as students engage in reasoning and justifying, their reasoning to others, they begin to develop proofs that are appropriate to that community.

2.2 A Problem Solving Environment

There is general agreement that mathematical reasoning and argumentation are often manifested through the process of mathematical communication in learning communities (Yackel and Cobb 1996; Forman, 2003). Communities of mathematical inquiry include students participating in mathematical discussions, proposing and defending arguments, and responding to the ideas and conjectures of their peers (Goos, 2004). According to McCrone (2005), this type of mathematical discourse increases when teachers and students share the responsibility of creating working norms and communicating about mathematical concepts.

Social norms for small-group activities include but are not limited to persistence in problem solving, explaining solutions to partners, listening and making sense of partner's explanations, and attempting to come to agreement on answers solutions processes. Social norms for the whole-class discussions often consist of explaining and justifying solutions, listening to and attempting to understand explanations of others, questioning each other, and sharing agreement or disagreement (Cobb, Yackel, and Wood 1995). Through the establishment of social norms, an effective mathematical community or micro-culture can be created. The mathematical community that includes established social norms as described above is often the impetus for sense making that leads to mathematical understanding. Through extensive analysis of an 18-year longitudinal/cross-sectional study, Other and Author (2005) found that mathematical reasoning was promoted when students were afforded opportunities to work collaboratively on strands of complex tasks and encouraged to take ownership of solution strategies and offer justifications. They suggest that by presenting students with a complex task, rather than scaffolding a set of simpler tasks, mathematical reasoning that leads to sense-making can evolve.

2.3 Tasks that Support Reasoning and Proof

Researchers note that a well-defined, open-ended task can provide the stimulus for reasoning as students are encouraged to explain and justify their ideas. Open-ended tasks are often defined as problems that offer students multiple options for solution strategies and at times have more than one correct solution (Hancock, 1995). Open-ended tasks often involve deeper thinking, making conjectures, defending solutions, and making generalizations (Kulm, 1994). Conversely, closed tasks questions involve responses from memory or the performance of rote procedures (Burns, 1997). Well-defined, open ended tasks can become the stage that allows students to make their ideas public especially when multiple representations and multiple strategies for solutions emerge (Author, 2002; Fransisco & Author, 2005). When used effectively, open-ended tasks promote higher-order thinking (Dyer & Moynihan, 2000). As students grapple with creating their own strategies and conjectures rather relying on memorization or given rules or procedures they develop "deep understanding" of the mathematics (Hiebert, Carpenter, Fennema et al, 1996). In addition, researchers note that tasks that are open-ended and challenging encourage students to rely on their own mathematical resources and make possible the building of new knowledge. They suggest, based on the results of detailed longitudinal studies of students' mathematical thinking, that affording students' sufficient time to work with each other with minimal teacher interventions, and time to revisit tasks and reflect on their prior work and the explanations of others tends to promote justifications and reasoning (Author, 2002; Author & Other, 1996).

Henningsen and Stein (1997) identify certain task features that promote student learning, including opportunities for multiple representations, multiple solution approaches, and mathematical communication. In addition, they suggest that making connections between students' prior knowledge and the ideas presented in the task can further promote higher level thinking. Lithner (2008) suggests that through task construction one can evaluate the type of reasoning that students will engage and often this prediction correlates with the reasoning students actively use.

In a study of how two middle-grade teachers supported student mathematical reasoning, Doerr and English (2006) found that in changing from an evaluative role to one of a listener, the teachers encouraged the students to create their own ideas and strategies. They denote the Self Evaluation Principle as the ability of students to evaluate their own solution strategies, and emphasize that this is a significant task feature in that it encouraged teachers to listen to student ideas. Thus students are given the opportunity to decide what is reasonable and revise their own ideas.

As indicated earlier, Other and Author (2005) contend that affording students opportunities to work on complex tasks, rather than scaffolding a series of simpler problems, enhances mathematical reasoning in students. In addition, rather than revisiting the same task, they recommend engaging students in strands of structurally similar and more complex problems over time. Engaging in strands of related problems supports students building of connections and stabilizes mathematical understanding of complex concepts. Findings from a longitudinal study show that when students were given the opportunity to work collaboratively, they began to question each others' arguments and ideas and ultimately developed a sense of ownership of the given tasks (Other & Author, 2005). This ownership can help students become more confident in their ideas, and can contribute to making the mathematics personally meaningful, an important prerequisite for building individual understanding.

Recent research in the area of task design that can elicit reasoning (Author, Author, and Author, 2010) indicated that very specific aspects of tasks encourage the elicitation of multiple forms of reasoning. This report, resulting from extensive analyses of the commonalities between reasoning on one fraction task implemented in the fourth and sixth grades, suggested that certain features of the task were crucial in the ultimate elicitation of indirect reasoning, reasoning using upper and lower bounds, and reasoning by cases. These features included non-existence and the use of finite sets, among others. Our present study differs from this research by examining the work of students that represent certain forms of reasoning across ages and from diverse student groups that were used in justifying solutions to tasks from both the fraction and counting strands of longitudinal and cross-sectional studies. The added challenge present in this analysis is that highly specific aspects of the tasks could not be readily identified, given the structural differences inherent in the tasks. However, since the tasks that are analyzed here were the catalysts of similar, varied forms of reasoning, our goal here was to identify somewhat more general characteristics of the tasks that may also be contributing factors in the consistent elicitation of varied reasoning. We present our method of analysis, detailed results, and a discussion of our findings below.

3. METHODOLOGY

3.1 Data Sets

The episodes presented in this paper come from three data sources[4]. The first is a yearlong study of students' mathematical thinking that was conducted by researchers in a fourth grade classroom in a suburban/rural school. These students

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were from a suburban/rural socioeconomic district and were nine to ten years of age. The class was composed of twenty-five students, fourteen girls and eleven boys. The researchers conducted interventions over the course of fifty sessions during the school year. Half of those sessions focused on the building of fraction ideas using manipulatives, including Cuisenaire rods.

The second source of data is from an informal after-school math program consisting of twenty-four twelve to thirteen year old students that was conducted by researchers in a low socioeconomic urban community, drawn from a school consisting of 99% Latino and African American students (see author 2007 for a more detailed analysis of data). The students voluntarily met twice a week during an after-school program to work on strands of open-ended math tasks. The first eight sessions focused on fraction activities involving Cuisenaire rods.

The third source is a longitudinal study, now completing its 21st year, in which students from a working-class community engaged in strands of mathematical investigations as a context for research on the development of students' reasoning and constructing of mathematical knowledge and understanding. This study followed a focus group throughout their K-12 students and beyond, and included a heterogeneous group of twelve students during grades 4-8. One strand of tasks that was introduced to these students focused on combinatorics, and took place during several sessions every year over the course of these students' elementary and secondary schooling. The data discussed in this paper was collected when the students were in fourth and fifth grade. Although the task was initiated during whole class and small group sessions, the results focus on group interviews of one to four students.

3.2 Procedure

In each study, the students were placed in heterogeneous groups; problems were posed and students were invited to explore solutions in their groups. The tasks were designed carefully, as task choice played an important role in the project's objectives. The researchers worked on the assumption that if the tasks were too simple, the students' schemas would not be enhanced, but, on the other hand, if they were too difficult, the students would not be engaged in finding solutions. In all three studies, students were encouraged to provide justification for their solutions and to challenge and question the explanations of others.

In order to support this, learning environments that fostered collective mathematical learning and individual growth were created. Similar characteristics of all three environments included: (1) engaging students in solving open-ended, tasks; (2) encouraging small group and whole class discussions; (3) encouraging students to share their own ways of thinking; (4) affording students' opportunities to defend their ideas and challenge the ideas of others; and (5) allotting sufficient time to explore and build understanding (Bauersfeld 1995; Author, 1996).

This paper reports on data from the first seven 60 minute sessions from the fourth-grade study and the first five 60-75 minute sessions from the sixth grade study. Data from the third study includes segments from sessions as fourth and fifth grade students investigated problems in counting and combinatorics.

3.2.1 Exploring fractions with Cuisenaire rods

The students in the first two groups worked collaboratively on tasks involving fraction relationships. Many of the tasks were identical in both studies. Students were offered Cuisenaire rods to build models of their solutions to the tasks. A set of Cuisenaire rods (see Figure 1) contains 10 colored wooden or plastic rods that increase in length by increments of one centimeter. For these activities, the rods are given variable number names and permanent color names.

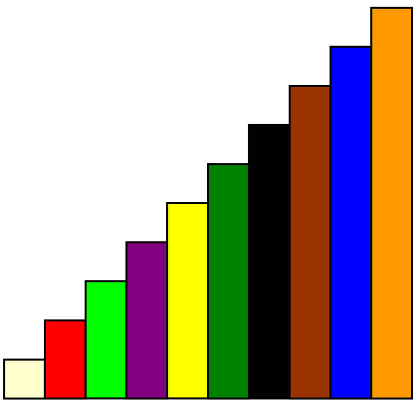

Figure 1. Staircase model of rods

A task that was posed early in both cycles is: “What number name would you give to the dark green rod if the light green rod is called one? Discuss the answer with your group”.

3.2.2 Exploring Combinatorics with Unifix Cubes

The students in the third source worked on building towers using plastic Unifex cubes selecting from two different colors (see figure 2). They investigated two similar tasks, one that required them to find all combinations of towers that were four cubes tall when selecting from two colors, and the other that asked them to find the same for towers that were five cubes tall. Over a period of three years, the students worked on these tasks in pairs and small groups and participated in whole class discussions as well as task-based interviews that focused on this set of tasks. In this intervention, as in the fraction interventions, the group investigations allowed students multiple opportunities to extend, revise, and share their solutions.

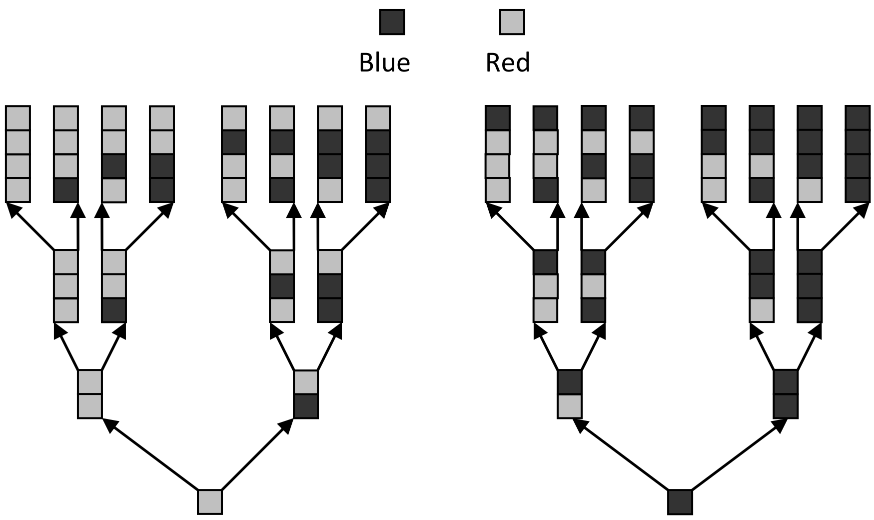

Figure 2. A tree diagram showing all possible towers 4-cubes high

3.3. Analysis

The primary source of data for this study was the database of video recordings that was created during each intervention. Analysis followed a modified grounded theory approach using widely accepted video data analysis methodology. Each of the three series of sessions was videotaped with at least two cameras. The video data were viewed, transcribed, and coded for critical events (forms of reasoning). The arguments were coded based on the form of reasoning used, including contradiction, cases, upper and lower bounds, recursive, and direct. Subcodes were developed for incomplete arguments and faulty reasoning. Then, the data sets were compared, and similarities and differences in the forms of reasoning used and the contexts in which they were elicited were noted (for an in-depth analysis of these sessions and a detailed record of the frequencies of each form of reasoning that was exhibited, see Author, 2007; Author, 2009; Author & Author, in press; Author, Sran, & Author, in press). Finally, the tasks were examined and analyzed according to what students were asked to produce and the forms of reasoning that occurred and connections were made across the data sources.

4. RESULTS

For the purpose of this paper, we focus on the two tasks described above. In the process of working on these tasks, students built arguments taking the form of both direct and indirect reasoning. In particular, for the fraction task, student argued using cases, contradiction, upper/lower bounds, and recursion. For the combinatorics task, reasoning took the form of cases, contradiction, and recursion. In addition, direct reasoning was flagged during the fraction tasks but was coded as faulty. No simple direct arguments were used correctly to support solutions while working on either task. Numerous examples of the above forms of reasoning have been documented (Other and Author, 1993; Author and Other, 1996, 2000: Other and Author, 2005; Author, in press; Author and Author, 2007, 2008). Here we offer representative examples that we regularly observed with a wide range of students from a variety of communities. Table 1 summarizes the representative sample of reasoning of each form of reasoning as it occurred across the three groups of students that will be the topic of this paper. Although these forms of reasoning were noted numerous times during each of the studies (for frequencies of occurrence of each form of reasoning, see Author & Other, 2009, Other, Other, & Author, 2010a, 2010b; Author, 2009), this paper focuses on one task from each study that successfully elicited a wide variety of reasoning from multiple students.

Form of Reasoning	Grade	Task	Student
Contradiction	4	Fractions	David, Alan, Jessica
	6	Fractions	Chris
	4	Towers	Stephanie
Cases	6	Fractions	Justina
	5	Towers	Stephanie
Upper & Lower Bounds	4	Fractions	David
	6	Fractions	Dante
Recursive	4	Fractions	Michael
	4	Towers	Milin

Table 1. Forms of reasoning that occurred across the three populations.

4.1 Reasoning by Contradiction

Reasoning by contradiction (also known as the indirect method; based on the assumption that whenever a statement is true, its contrapositive is also true) was used by all three groups of students. When working on the fraction task, contradiction was used to convince their classmates that there was not a rod whose length was half of the blue rod and that they had built all of the towers of a given (n) height. When investigating the towers task, this form of argument was used to show that all towers of a specific case were accounted for.

4.1.1 Fractions, Grade Four

After a student suggested that the yellow rod and the purple rod could each be called one half if the blue rod was called one, David used reasoning by contradiction to explain in order for each of two rods to be called one half the length of the blue rod, the two rods would need to be the same length. He then showed that the two rods in question differed in length He used a model of a purple rod and a yellow rod placed next to a blue rod and argued using the definition of one-half, explaining that in order to be called half of the blue rod the two rods would need to be the same length. Alan and Jessica built on David's argument and together formulated a contradiction.

Alan: When you're dividing things into halves, both halves have to be equal – in order to be considered a half.

Jessica: [inaudible] this isn't a half. Those two aren't both even halves.

4.1.2 Fractions, Grade Six

In the sixth grade, Chris reasoned using a contradiction and lined up a train of nine white rods next to a blue rod, showing that the blue rod was equivalent to an odd number of white rods, and that it could not be divided in half. Chris explained:

> *There is not a rod that is half of the blue rod because there's nine little white rods, you can't really divide that into a half, so you can't really divide by two because you get a decimal or a remainder…*

4.1.3 Towers, Grade Four

Stephanie approached the task of finding how many towers (height of five) could be built by applying the procedure of constructing a tower and it's "opposite" to find the 32 unique towers that were five cubes tall. Stephanie explained how she knew that she had accounted for all the ways that towers could be formed from two red cubes and three yellow cubes:

> *With the two [red cubes] together you can make four [towers]. With one [yellow cube] in between you can make three [towers]. With two [yellow cubes] in between you can make two [towers]. With … three [yellow cubes] in between you can make one [tower], but you can't make four in between or five in between [four or five yellow cubes between the two red cubes] …or anything else because you don't have enough … because you can only use five blocks [towers of height five].*

4.2 Reasoning by Cases

For the purpose of this study, critical events were coded as reasoning by cases when students defended an argument by defining separate instances and discussing the implications or inferences drawn from each instance.

4.2.1 Fractions, Grade Four

David offered an argument by cases to show that all of the rods could be organized as either odd or even based on whether or not they could be divided in half. He explained that the white, light green, yellow, black and blue rods were all "odd" since there was not a rod equal to half of their length. He then showed that the red, purple, dark green, brown, and orange rods were "even", using a model to show that two purple rods are equivalent to the length of the brown rod and two yellow rods are equal to the length of the orange rod (in order to demonstrate that these rods had a half). The overhead transparency view of David's model is shown in Figure 3.

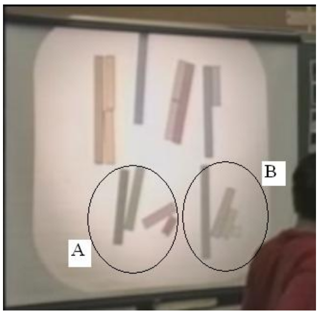

Figure 3 David's categorization of (A) even and (B) odd rods.

4.2.2 Fractions, Grade Six

Justina explained that her strategy of showing that the blue rod does not have a rod that is equivalent to half of its length was to instead find all of the rods that do have a rod equal to half of their length. She drew all of the rods that have a half next to the two identical rods that could each be called one half. For example, she lined up two yellow rods next to an orange rod, and did the same for all other rods of its kind. She named this set of rods "singles".

4.2.3 Towers, Grade Five

Stephanie explained how she organized her groups of towers (4-high) according to color categories (e.g., exactly one of a color and exactly two of a color adjacent to each other) in order to justify her count of 16 towers (see figure 2). She showed how she accounted for all possible color combination of each color category thereby accounting for all cases of towers with a height of four.

4.3 Reasoning using Upper and Lower Bounds

When reasoning using upper and lower bounds, students defined a set by the outer limits of the set, and talked about the elements of the set that were contained within the bounds, and about elements that were not part of that set because they were larger than the upper bound or smaller than the lower bound that was defined. When reasoning about the rods, the students defined the set as containing one element - that which was half the length of the blue rod. They then showed that none of the rods were contained within their specified bounds, thus proving that no rod existed that was half the length of the blue rod.

4.3.1 Fractions, Grade Four

David began the task of convincing his classmates that there was not a rod whose length was half of blue by offering an argument using upper and lower bounds, explaining, “I don’t think that you can do that because if you put two yellows that’d be too big, but then if you put two purples that’s uh, that’s uh, that’d be too short.” He then showed that the purple rod was one white rod shorter than the yellow rod, and lined up the rods in a staircase pattern in order to illustrate that each rod was one white rod longer than the previous rod. He used this model (shown in figure 4) to show that there is no rod that is shorter than the yellow rod or longer than the purple rod.

Figure 4 David’s model for his argument using upper and lower bounds

4.3.2 Fractions, Grade Six

Dante explained that the combination of two purple rods was too short to be equivalent to half of blue and the combination of two yellow rods was too long. He further explained that the yellow rod was one white rod too long and the purple rod was one white rod too short.

4.4 Reasoning Using a Recursive Argument

Recursive reasoning was noted when a student used the repetition of a basic case or operation to define and discuss a class of objects, and then used this repetition to explain what a complex case would look like or to show that a calculation was impossible.

4.4.1 Fractions, Grade Four

After the students determined that there was no rod that was half the length of the blue rod, the class then discussed the possibility of "cutting" a rod in half to create a new rod and therefore finding rods that were one half the length of "odd" rods. Michael explained, "If you're going to make a new rod, then you'd have to make a whole new set because there'd have to be a half of that rod, too". David reinforced Michael's argument by explaining that each time a smaller rod was cut in half, it's half would have to be cut in half and therefore a new set would emerge.

4.4.2 Towers, Grade Four

Milin's explanation was based on building from a shorter tower exactly two towers that were one cube taller. For example, when asked to explain why from two towers he created four, Milin pointed to his towers that were one cube high and explained,

> *"Because – for each one of them, you could add one – No – two more – because there's a black, I mean a blue, and a red- See for that you just put one more – for red you put a black on top and a red on top – I mean a blue on top instead of a black. And blue – you*

put a blue on top and a red on top – and you keep doing that and for each one you keep on doing that and for 6 you'd get 64".

5. Discussion

Our results indicate that both tasks, one focusing on fraction ideas and the other on combinatorics, elicited similar forms of reasoning at multiple grades levels across different socioeconomic communities. While attending to both tasks, students' arguments took the forms of reasoning by contradiction, cases, using upper and lower bounds, and by recursion. Although one of the highlighted tasks focuses on fractional relationships and the other on combinatorics, specific characteristics of these tasks may have elicited these complicated forms of reasoning.

5.1 The Fractions Task

In this task students were challenged with finding a non-existent rod. The open-ended nature of this task led to more complicated forms of reasoning. David and Chris built a contradiction based on the definition of one-half. Justina used a pattern to organize the two types of rods (those with and without a half) into cases in order to make sense of why the blue rod did not have a rod equivalent to half of its length. David and Dante found the most likely rods to equal half of the blue and created the upper and lower bounds. Finally, in trying to make sense of an endless task, Michael and David showed the endless possibilities.

5.2 The Towers Task

When attending to the towers tasks students were looking at many variations of the task and trying to find a pattern to use for organization. Stephanie based her argument on the contradiction of the definition of the task (5 cubes tall). Stephanie then used patterns to reorganize her towers into cases in an effort to account for all possibilities. Milin found a formula to use to calculate combinations for ever increasing tower heights.

5.3 Characteristics of Both Tasks

While one task focused on fractions and the other combinatorics, and the tasks seem quite different on the surface, both tasks encouraged students to use various forms or reasoning and share specific characteristics that may help explain why the particular forms of reasoning emerged. In both tasks, students were invited to use manipulative materials but could also approach the tasks with other representations. The building of models naturally led to student collaboration. Students were engaged in discussion as they worked together to build these models. The models that they built required understanding of the problem and they worked together to achieve this understanding. The varieties of representations that the students used to express their ideas were shared in open discussion with others, providing opportunity to build on each other's ideas.

Both tasks were open-ended, challenging, and allowed for multiple entry points. They were open-ended and non-routine, in that a solution was not readily available and students were expected to rely on their own resources to solve and justify their solutions. This meant that students at different levels of mathematics knowledge could all engage in the problem tasks and achieve success. Due to the nature of these tasks, students had opportunities to extend their understanding and communicate their ideas in the arguments they built to support their solutions. In addition, both tasks were open to multiple representations and multiple strategies for solutions. These multiple strategies elicited various forms of reasoning, as in the case of David, who justified his reasoning using four different forms of reasoning when attending to the fraction task. Moreover, students had the opportunity to revisit the tasks; this provided further opportunity to reflect on their previous ideas and arguments as well those offered by their classmates. Thus, when revisiting the tasks students had earlier knowledge on which to build and often revised and/or extended their original ideas.

Due to the complex nature of the tasks, students were not able to form arguments using simple direct reasoning. In both fraction groups, students attempted to reason directly but created faulty arguments. This was due to the nature of these tasks, which required students to build arguments that were more detailed and contained other forms of reasoning. Other more closed tasks used in the teaching interventions elicited less of a variety of reasoning. For example, during the third session of the sixth grade intervention, the students were given the task, "What is the number name for red (when blue is named one)?" All of the groups used direct reasoning based on the relationship between the red, white, and blue rods to name the red rod two-ninths. During the last four sessions of the fourth grade intervention the tasks focused on division of fractions. Few counterarguments and indirect arguments were noted during these sessions. Only direct arguments were given during the last two sessions, and the two preceding those were marked by only five indirect counterarguments in total. For example, the students were asked to name the white rod when the dark green rod was called one, and then were asked how many white rods are in the dark green rod. They then discussed how to write a number sentence using that information ($1 \div 1/6 = 6$). These closed tasked were could be attended to using straightforward procedures and therefore required students to rely on their own thinking and input from their partners to solve and justify their solutions.

We contend that task design must be approached at multiple levels. In addition to the precise framing of the task and consideration of structural features of tasks that encourage reasoning (Author, Author, & Author, 2010), we suggest following our present analysis that careful attention must be paid to the presentation of the task, as well as the variety of learning opportunities that the task presents for students. In the tasks analyzed here, the tasks were designed with the intention to allow students to maximize their learning opportunities to the fullest, and were presented, often multiple times, in a way that actualized that intent, in the form of the encouragement of varied reasoning. It is important to note that the tasks used in the study were not designed to elicit these forms of reasoning; rather the study aimed to learn about children's general mathematical thinking. The variety in the forms of reasoning that was later observed, therefore, was not planned in the initial design. The commonalities that were observed across the three very different studies, therefore, lend credence to the hypothesis that the tasks, as well as the method of their implementation, may share characteristics that can be replicated to produce similar results in diverse student populations. Although the characteristics of the tasks that are discussed here may not have contributed equally to the elicitation of varied reasoning, they are all possible contributing factors that are worthy of further, more detailed study.

We suggest that strands of tasks such as these be integrated into regular mathematics instruction and that students be asked to revisit the same or similar tasks, so that they can build on and extend their approaches, offering opportunities to experience a variety of ways of reasoning. Tasks such as these can serve to engage students in doing mathematics and building arguments. Knowing in advance the types of tasks that elicit varied forms of reasoning can assist teachers in the goal of promoting reasoning at all grade levels. We suggest that strands of open-ended tasks that elicit these forms of reasoning be integrated in the curriculum at all grade levels.

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