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Conics Studium 3D: Integration of Computational Resources to Articulate the Semiotic Representation Records of Conic Curves

Conics Studium 3D: Integração de Recursos Computacionais para Articular os Registros de Representação Semiótica de Curvas Cônicas

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Abstract

Conics have been studied by mathematicians over the centuries, especially through the different models of geometry: synthetic (Greek), analytic and projective. However, the study of conics in basic education is extremely limited whether in Brazil, France, Spain or Mexico, and although it is still present in several university courses, their approach is fragmented and the focus is on algebraic representations. In order to cope with the difficulties encountered in teaching conic curves, we propose the design of a computer artifact. We show the need to favor an integration of synthetic and analytic approaches, as opposed to their fragmentation, an integration that considers an articulation between the different representations, linking the visual variables to the corresponding symbolic units, via transformations by conversion and processing using a computer resource. To meet these requirements, beyond the mathematical study and its history, we undertook a cognitive study to specify the dynamic articulation of various conic curves representations by means of computer resources, so that the artifact designed can contribute to the overall development of reasoning, analysis and visualization abilities of students and teachers. For validation, we have developed the 3D prototype of Conics Studium, offering the possibility to explore what we call "dynamic semiotic registers", that is to say to dynamically articulate equations (and their different forms - transformation by processing), flat graphical representation and spatial graphical representation of conics, so that by making displacements in the spatial figurative representation, we obtain a modification of the Cartesian graph and algebraic expressions, allowing a fast and continuous access to several situations (Transformation by conversion).
Keywords: Conic Curves. Semiotic Representations. Dynamic Semiotic Record. Educational Software.

Resumo

As cônicas têm sido estudadas pelos matemáticos ao longo dos séculos, principalmente através dos diferentes modelos de geometria: sintética (grega), analítica e projetiva. Porém, o estudo das cônicas na educação básica é extremamente limitado seja no Brasil, na França, na Espanha ou no México, e embora ainda esteja presente em diversos cursos universitários, sua abordagem é fragmentada e o foco está nas representações algébricas. Para dar conta das dificuldades encontradas no ensino de curvas cônicas, propomos o projeto de um artefato computacional. Mostramos a necessidade de favorecer uma integração das abordagens sintéticas e analíticas, em oposição à sua fragmentação, uma integração que considere uma articulação entre as diferentes representações, ligando as variáveis visuais às unidades simbólicas correspondentes, através de transformações por conversão e processamento usando um computador recurso. Para atender a esses requisitos, além do estudo matemático e de sua história, empreendemos um estudo cognitivo para especificar a articulação dinâmica de diversas representações de curvas cônicas por meio de recursos computacionais, para que o artefato desenhado possa contribuir para o desenvolvimento global do raciocínio, análise e habilidades de visualização de alunos e professores. Para validação, desenvolvemos o protótipo 3D do Conics Studium, oferecendo a possibilidade de explorar o que chamamos de "registros semióticos dinâmicos", ou seja, articular dinamicamente equações (e suas diferentes formas - transformação por processamento), representação gráfica plana e representação gráfica espacial de cônicas, de modo que fazendo deslocamentos na representação figurativa espacial, obtemos uma modificação do gráfico cartesiano e das expressões algébricas, permitindo um acesso rápido e contínuo a diversas situações (Transformação por conversão).
Palavras-chave*: Curvas Cônicas. Representações Semióticas. Registro Semiótico Dinâmico. Software Educacional*

1 Introduction

The conic sections have been the subject of investigation over centuries from various perspectives, including synthetic geometry (from Greek), projective geometry, and analytical geometry. However, the curriculum in Basic Education in countries like Brazil, France, Spain, and Mexico, for example, is extremely limited concerning conic sections. Although these topics continue to be present in various university courses, such as Analytical Geometry, Calculus, Differential Geometry, and Graphic Geometry (both two-dimensional and three-dimensional), their teaching often happens in a fragmented and superficial manner. Analytical approaches with an algebraic focus are often prioritized.

The fragmented approach to teaching conic curves has led to difficulties not only in Brazil (Santos Neto, 2006), (Chagas, 2017), (Quaranta Neto, 2013), (Bordallo, 2011) and France (Trgalová, 1995) but also in Spain (de la Fuente, 2002), (Gascón, 2002) and Mexico (Velázquez, 2007), as indicated by various studies.

When analyzing the construction of knowledge about conic sections, efforts have been made to articulate their definitions, identify different aspects, and construct necessary mathematical tools. In this process, it became evident that conic sections could be characterized through at least three types of approaches: a synthetic approach, where curves are obtained by intersecting a circular cone with a plane; an analytical approach, characterized by quadratic equations (general or reduced forms), matrix representation, and graphical representation on a coordinate system; and a projective approach, where conic sections can be defined as projections of circles onto a plane.

Trgalová (1995) points out a significant difference between the place of this notion in scientific knowledge and in school knowledge. She adds that the difficulties in teaching conic sections arise from the complexity and richness of the mathematical object as well as the breadth of geometric knowledge required to understand it.

To address these challenges, it is necessary to adopt a cognitive approach that facilitates the dynamic integration of various representations of conic sections through computational resources. This approach aims to enhance mathematics education by contributing to the overall development of students' and teachers' reasoning, analysis, and visualization skills.

In this context, we believe that conic sections should be studied as a unified object, including Apollonius' definition and the Dandelin-Quetelet theorem (synthetic geometry), definitions through quadratic equations (general and reduced forms), matrix representation (analytical geometry), and Pascal's and Brianchon's theorems (projective geometry) as auxiliary representations in transitioning between algebraic and spatial representations.

To overcome these challenges, our preliminary studies led to the development of a computational artifact based on the following principles:

- Epistemological Study: Emphasizing the importance of examining conic sections from various geometric approaches and vice versa, considering synthetic, analytical, and projective geometries. Articulating or studying equivalences between definitions and properties of conic sections should consider these diverse approaches.
- Cognitive Perspective: Highlighting the importance of integrating various representations of conic sections (algebraic, Cartesian plane figurative, and spatial) and their specific elements (foci, directrices, eccentricity, vertices, axes, among others). This integration is especially crucial by utilizing computer technology to create dynamic representations and integrations considering visual variables (foci, directrices, eccentricity, vertices, and axes, as shown in the diagram) and visual figures (foci, directrices, vertices in the spatial perspective figure).
- Didactic Analysis: Recognizing the need to integrate different geometric approaches to conic sections and to orchestrate situations (and artifacts) that facilitate these approaches.

It is important to note that the elicited requirements, considering epistemological, cognitive, and didactic aspects, contributed not only to the conception and development of the software, which will be discussed further, but also to the analysis of existing programs that could potentially meet the demands.

2 Development

2.1 Educational software characterizations

It is worth noting that the teaching of conic curves using computational resources has been the subject of various studies. In light of this, we analyzed the functionalities and possibilities of dynamic geometry software Cabri-Géométre and GeoGebra in articulating representations of conic curves in fourteen research studies.

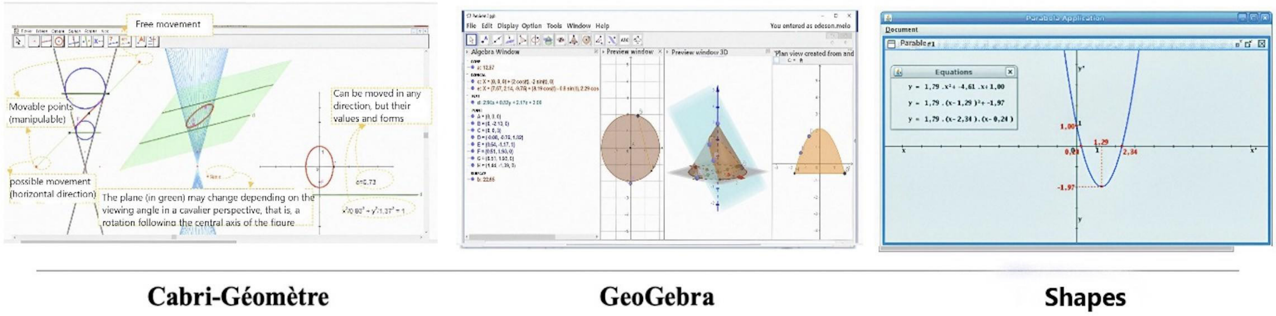
However, we found that Cabri-Géométre demonstrated limitations, even though it allowed the creation of simulations. Due to excessively laborious constructions, it compromised the usability of the software in dynamically articulating conic representations. GeoGebra, on the other hand, did not meet the needs of our study, especially concerning the integration of visual variables with corresponding symbolic units. Specifically, the key issues that disqualified the analyzed software and guided our development related to the functionality of algebraic expressions and the lack of simultaneous articulation of the three representations.

It is important to highlight that the *Formas* (Shapes) application presents some functionalities that seem to meet the requirements for dynamically articulating conic representations. However, it only addresses the articulation between algebraic expressions of quadratic equations and their graphs, specifically parabolas.

Since we lack a resource similar to *Formas* for conics, enabling the exploration of their dynamic semiotic representations - dynamically articulating their equations (and their various forms), their graphs, and their spatial figural representation, allowing modifications in algebraic forms and spatial figures as the graph is manipulated - we propose the development of a new application. This application should facilitate the direct manipulation and articulation of representations, particularly for conic curves. It will function as an integrated and dynamic representation system, guiding the user in constructing knowledge by exploring object behaviors and phenomena, the variety of records, and definitions.

In analyzing existing software, we identified limitations such as the inability to articulate spatial figural representation along with algebraic and Cartesian graphical representations. Regarding the symbolic representation in algebra, one of the most specific elements we investigated that disqualified the analyzed software and directed our development pertains to the functionality of algebraic expressions. In summary, the introduction of conic curves articulating all three representations simultaneously is not supported by current software.

Figure 1 - Interfaces of the three analyzed software



Source: Siqueira (2009)

We emphasize that the elicited requirements serve to formulate and substantiate the principles of the most suitable software for conic studies (Figure 1 illustrates all 3 afore mentioned softwares). They also help us analyze the state of the art and potentially identify elements not described in the requirements but that can align with them.

2.2 Prototyping

Our prototype development proposal considered forming a team with members from different fields of knowledge, allowing them to contribute their expertise to the product being created. The multidisciplinary team consisted of members from the LEMATEC Research Group and students from three distinct undergraduate programs at the Federal University of Pernambuco. These three collaborating members played a fundamental role throughout the process, from conception to development. They contributed their expertise and brought their studies and research results into the discussion to assist in the investigation, as shown in Table 1.

Table 1 - Problems for prototype development and professionals to solve them

Problem(s) to solve	Professional to solve it
Team management and technical-scientific coordination	Graduate in Mathematics and Master's degree with research in Educational Computational Technology.
Apply principles and theoretical and methodological foundations of projective geometry to conic curves. Establish guidelines for the creation and development of the interface (definition of elements of interaction between spatial figural and Cartesian graphical geometric representations, enabling the definition of parameters that structure the algorithm used to facilitate articulation between representations). Enhance the prototype visually while considering its functionalities.	Content expert, holder of a degree in Graphic Expression, or currently pursuing a degree in Graphic Expression with knowledge in graphic design.
Develop the 3D modeling project, performing programming and graphic editing of algorithms	Expert in parametric design and modeling, with knowledge in graphic representation software and experience in Grasshopper.
Develop the prototype focusing on programming issues involving Grasshopper, such as plugin integration, using Python and C# scripts to generate mathematical equations (algebraic representations whose coefficients are corresponding symbolic units) used to create the geometric elements of conic curves (geometric representations where the visual variables are the elements).	Specialist in system development.

Source: research data.

It is important to highlight that the specified requirements, such as simultaneous articulation between different representations, using any representation as an input method through direct manipulation of geometric representations (2D graphical and 3D spatial figures) and editing algebraic expressions, required guidelines for programming, graphic algorithm editing, and the creation and development of the interface and 3D modeling. Initially, we intended to develop the prototype using JavaScript as the programming language. However, due to the unavailability of a library that would allow more agile development to meet the prototype requirements, we chose to use a program that, although not specific for this purpose, could give the collaborating members (participating teachers in the validation) an idea of the software's functionalities we intended to develop. Thus, we implemented algorithms in the Rhinoceros software (version 5) using the Grasshopper® plug-in (version 0.98).

After choosing the programs, we focused on the prototyping process and implementing functionalities based on the specified characterizations and requirements.

Conics Studium 3D: Articulating Dynamic Semiotic Records of Conics

Conics Studium 3D emerged from the study and the absence of software that met the demanded needs, namely, the requirements to develop a computational environment capable of integrating various representations of conic curves. Consistently and interconnectedly, these three representations should have a single data source, meaning that all three representations present the same object in three different ways.

It also offers the opportunity to explore what we term as "dynamic semiotic records", that is, dynamically articulating equations (and their various forms), their Cartesian graphical representation, and their spatial figural representation. By making changes in the spatial figural representation, modifications occur in the Cartesian graph and algebraic expressions, enabling quick and continuous access to various situations.

We will briefly discuss what the built prototype of Conics Studium 3D can provide through transformations by handling algebraic representations and, conventionally, spatial figural representation, integrated with Cartesian graphical representation and algebraic expressions, always considering spatial figural representation as the input record.

In the Conics Studium 3D interface, we can see three windows, one for each representation of conics, and the remote-control bar, with selectors to choose: the type of curve, the inclination angle of the cutting plane, the cone's height, the parametrized coordinates of five points, coefficients of the general quadratic equation, a multiplicative factor, the general and reduced equations, and the unit determining the grid spacing.

2.3.1 Spatial Figural Representation in Conics

The Conics Studium 3D prototype presents, in a synthetic geometry approach, a circular cone of revolution intersected by a plane, as shown in Figure 2. The figural visual variables of the spatial figural representation pertain to the elements of the cavalier perspective representation of the cone intersected by a plane generating the conic section. The intersection of these elements and other subsequent elements generates a conic section with its focus or foci (points) and its directrices. For instance, all these elements are generated from the tangency between the Dandelin spheres and the section plane(s) and between this and the directrix plane(s), respectively. These elements are related to the visual variables of the Cartesian graph and the parameters of the equations. Other elements, such as vertex or vertices, and five distinct points belonging to the curves, can also be mentioned. Moreover, we were able to implement ideas from Apollonius' definition, enabling all conic sections to be defined in the same cone.

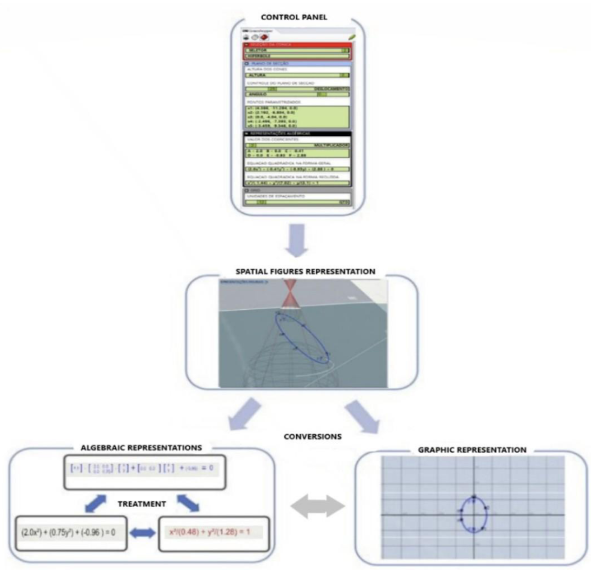
In this window, we can explore the foci, the directrices, five arbitrary points that satisfy the conic section, the angle of inclination of the section plane, and the height of the cone. Another possibility is obtaining the coordinates on the Cartesian plane of some points on the conic sections (foci and arbitrary points). This feature takes into account the dimensional deconstruction of the cone intersected by a plane, for example, in 3D, and the conic curve, in 2D, and auxiliary traces, where we can identify figural units of smaller dimensions (1D and 0D) such as lines and points.

Algebraic Representations in Conics

In the conversion from the spatial figural representation of a conic section to the Cartesian graph and its algebraic expressions, we consider auxiliary transition representations, that is, the results of Pascal's theorem and Brianchon's theorem for conics, ensuring that five non-collinear points determine a conic curve. To achieve this, we take the coordinates of the five points and through the solution of a 5x5 system (obtained from the coordinates of each point), we obtain the coefficients of the general quadratic equation of a conic section. Through further treatments, we derive the matrix form and the reduced quadratic equation.

Thus, the algebraic representations window presents three equations for a conic curve, namely, the general quadratic equation, the canonical equation, and the matrix form. Below, we present three examples of algebraic representations available in Conics, one for each conic curve.

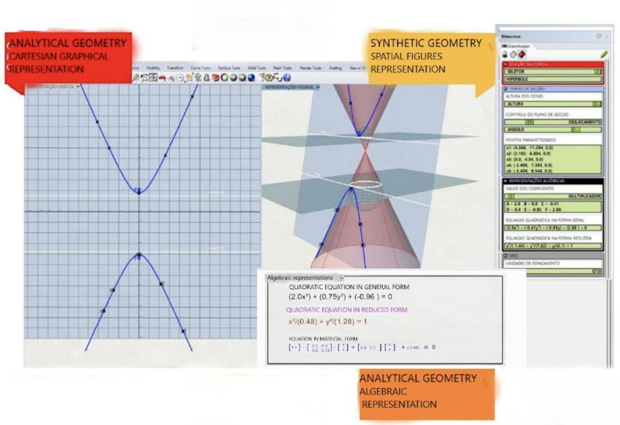
Figure 2 - Spatial Figural Representation of the Geometric Record with input from the control panel.



Source: Siqueira (2009).

With this, the software presents the representations considering the significant characteristics (values and coefficients' conditions) of algebraic expressions so that the Cartesian graph respects a global, dynamic, and simultaneous understanding, as shown in Figure 3.

Figure 3 - Highlight of the hyperbola defined from synthetic and analytic approaches in Conics Studium 3D



Source: Siqueira (2009)

The interface (or field of view) of Conics Studium 3D shows successive displays of simultaneous variations of responding symbolic units (coefficients) from three algebraic representations linked to the visual units of the graph and figural visual representation in space.

We can observe that the reduced equation displays the values of its coefficients in the control panel. Additionally, the matrix and quadratic equations in the general form justify the conversion from spatial figural and Cartesian graphical representations to algebraic expressions, considering any five points of the conic section.

Another relevant aspect concerns the fact that conic sections are also algebraically constructed via monofocal definition (one focus, the directrix, and an arbitrary point), implying that we have at our disposal the same characteristic elements of conics both in the form of visual variables (foci, directrices, vertices) and in the corresponding symbolic units (coefficients of equations).

Addressing the articulation between the representations of conics in this perspective corresponds to a systematic description process of visual variables, considering the global interpretation procedure of Cartesian graph properties and spatial figures. For instance, the trace/axis set forms a graph representing the object described by algebraic expressions, allowing the identification of modifications made in graphical representation and algebraic expressions.

In the Conics interface or field of view, successive displays of simultaneous variations of figural visual variables spatial representation, corresponding symbolic units (coefficients) from three algebraic representations linked to the visual units of the Cartesian graph, appear. In our prototype, the input record is made from spatial figural representation, and the output records can be the Cartesian graph or algebraic expressions, depending on the teaching situation's needs.

Thus, Conics Studium 3D interface consists of three representation windows and a remote-control panel.

The remote-control, serving as the input for spatial figural representation, replaces the direct manipulation of objects. Here, we can select the type of conic section, the height of the cones, and the inclination angle of the section plane. Concerning the Cartesian graphical representation, we can choose the grid spacing unit. As for algebraic representations, considering that Rhinoceros/Grasshopper is not a mathematical program, we provide the option to select a scalar to facilitate rounding decimal values.

2.4 Elicited and Implemented Requirements

The alignment between the Epistemological, Cognitive, and Didactic dimensions, which pointed towards the Informatics dimension, aimed to establish fundamental principles to guide the characteristics and functionalities that were subsequently elicited and, for the most part, implemented in the software, as observed in Table 2.

Table 2 - Requirements elicited in the epistemological, cognitive, and didactic dimensions and implemented in the Conics prototype. Source: Siqueira (2009)

Dimensions	**Requisites**	
	Elicited	**Implemented**
Epistemological	To articulate characteristic elements and representations in the construction of the concept of conic sections, as present in synthetic, analytic, and projective approaches, using computational resources.	Integrate elements and characteristic representations in constructing the concept of conic sections, present in synthetic and analytic approaches.
Cognitive	Represent conic sections from algebraic, Cartesian graph, and spatial figural perspectives; Simultaneously articulate the representations through treatments (algebraic expressions) and conversions (equations, graph, and spatial figures); Dynamically articulate the representations of conic sections through computational resources, involving a "dynamic semiotic record".	Represent conic sections from algebraic, Cartesian graph, and spatial figural perspectives, simultaneously articulating the representations through transformations involving treatments (algebraic expressions) and conversions (spatial figures, Cartesian graph, and equations).
Didactic	Dynamically articulating the semiotic representation records of conic sections through computational resources aims to meet the needs for their effective teaching. For example, integrating approaches considering their definitions via focus, directrix, and eccentricity, connecting their geometric and algebraic objects (addressing issues of fragmentation with an algebraic focus).	Dynamically articulate the semiotic representation records of conic sections, integrating various approaches considering their definitions via focus, directrix, and any five points, connecting their geometric objects (visual variables: focus, directrix, vertices, and any five points) and algebraic aspects (corresponding symbolic units).

Source: research data.

From the perspective of the informatics dimension, Conics Studium 3D met most of the essential characteristics and requirements to address the specificities derived from the studies of the other three dimensions, as we can observe in Table 3.

This computational environment facilitates the integration of various artifacts, the manipulation, and articulation of representation systems, in the case of conics. We believe that an integrated and dynamic representation system can guide the user in constructing knowledge through exploring the behaviors of objects and phenomena, utilizing a variety of records and definitions. Therefore, we hope that some of the difficulties identified in our research related to teaching conics and integrating different algebraic, Cartesian plane, and spatial figural representations can be overcome.

Table 3 - Characterizations and functionalities implemented based on the elicited requirements from the informatics dimension

Conic Studium 3D		
Dimensions	**Elicited**	**Implemented**
Informatics	Metarepresentational environment capable of dynamically representing and articulating various conic section representations simultaneously, constituting what we refer to as a "dynamic semiotic record"; Integration of resources dealing with synthetic, analytic, and projective geometries and their corresponding representations; The interface, or field of view, presents successive displays of concurrent variations of corresponding symbolic units (coefficients) from three algebraic representations, linked to the visual variables of the Cartesian graph and the figural spatial representation.	Simultaneous articulation of algebraic representations, Cartesian graphs, and spatial figural representations, with input registration through the latter; Explication and articulation of at least two algebraic representations; Display of successive algebraic variations (among all possible expressions), Cartesian graphs, and spatial figural representations on the interface; Enables interactions; simulations; visualizations; and explorations of algebraic and geometric objects through a control panel.

Source: Siqueira (2009)

We emphasize that difficulties associated with eccentricity definition cannot be addressed since the prototype does not provide any operation in that regard. Furthermore, recognizing conic objects when their vertices are not on the ordered axes cannot be explored either, as the application allows only situations involving Cartesian graphical representation with foci on the ordered axes.

Hence, our aim was to ensure that Conics met the characteristics of conic objects and their representation, intending to enhance their teaching, as shown in Table 3. This was done through analyses of the epistemological, cognitive, didactic, and informatics dimensions.

2.5 Applications of the Conics Prototype to Certain Difficulties Related to Teaching Conic Curves

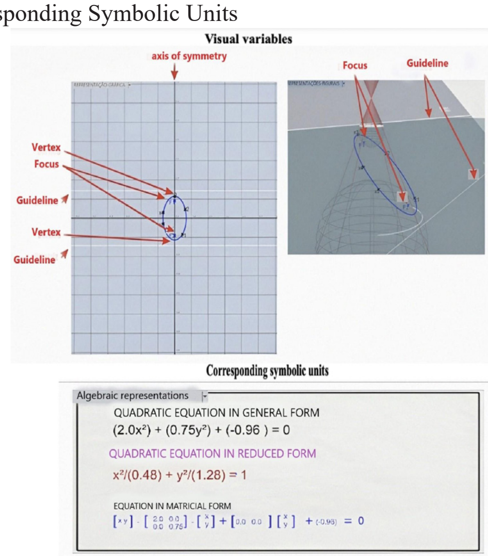
The functionalities of the Conics prototype can be exemplified by considering transformations through conversion and the challenges in articulating different representations and transformations through treatment, as well as difficulties in coordinating the same record. In this way, we will demonstrate how some difficulties can be explored using this computational tool.

2.5.1 Transformations through Conversion and Challenges in Articulating Different Representations

The capability to make simultaneous alterations between visual variables and corresponding symbolic units provided by the Conics prototype may facilitate addressing the types of difficulties highlighted in this research, particularly the challenge of associating the corresponding symbolic units (equation coefficients) with the visual variables in Cartesian graphical representations (geometric objects such as foci, vertices, directrices, etc.).

As we can observe in Figure 4, the visual variables common to the Cartesian graphical and spatial figural representations of the ellipse (such as foci, vertices, directrices, for example) related to the corresponding symbolic units, namely, the parameters *a* and *b* from the reduced equation, as well as the coefficient *c*, provide information about the curve's global characteristics and properties. For instance, they reveal the positions of the directrices below and above the vertex (with the values of *a* and *b*), foci above and below the x-axis (knowing the value of *c*), or vertices on the minor axis, to the right and left of the focal axis (with the value of *c*).

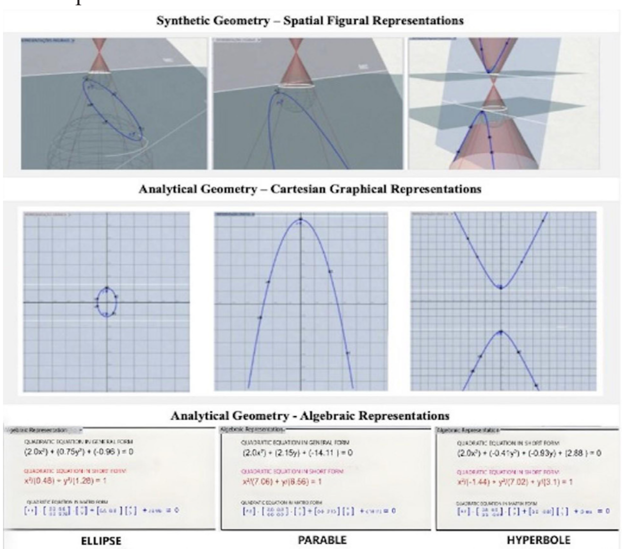
Figure 4 - Correspondence between Visual Variables and Corresponding Symbolic Units



Source: Siqueira (2019, p.243)

The visualizations provided by the Conics prototype, combined with the ability to observe the curve as a section of double cone based on elements of a curve plotted in the coordinate system on the Cartesian plane, for instance, can facilitate the transition from algebraic to graphical representation, or vice versa. For example, it can decode a geometric representation, discover the structural relationships, and even describe them while establishing a connection with the coefficients of algebraic expressions, as shown in Figure 5.

Figure 5 - Integration between two approaches to geometry and three representations



Source: Siqueira (2019).

This integration between synthetic and analytical approaches and their respective representations - spatial figural, cartesian graphical, and algebraic - also facilitates overcoming difficulties related to transitions between Cartesian graphical representation and algebraic representations, and vice versa; between algebraic representations and Cartesian graphical representation, as well as between geometric records (Cartesian graph and spatial figural) and algebraic representations.

In this sense, we emphasize that using the same structure in the three windows, such as the cone, the section plane, Cartesian plane, and the three algebraic forms, to represent each of the conic curves, integrating their various representations, aids in the perception that the three types of conics are structurally related, directly or indirectly. Furthermore, the overall understanding of the curve's characteristics through the articulation between visual variables and corresponding symbolic units helps sketch the conic curve given one of its algebraic representations, especially since students often struggle with this transition, commonly resorting to the "point-to-point" method.

2.5.2 Transformations through Treatment and Challenges in Coordinating within the Same Representation

The explicit representation of three algebraic forms - the general quadratic equation, the reduced form, and a matrix form - interconnected with each other, can facilitate the execution of transformations through treatment of algebraic representations. This difficulty is linked to the fact that students often recognize only one form, limiting their ability to establish correspondence between visual variables and symbolic units, as they tend to ignore other algebraic forms.

On the other hand, the challenge associated with recognizing the nature of the curve, such as understanding the generation of curves as sections of a cone, leading to confusion between the parabola and the hyperbola, the latter being falsely interpreted as a double parabola, can be addressed using the window focusing on spatial figural representations. This approach enables the exploration of obtaining the three curves from a double cone, intersected by a plane, with corresponding changes in algebraic and Cartesian graphical representations. In this scenario, each conic curve exhibits its distinctive properties.

3 Conclusion

In general, we understand the role of the prototype in fulfilling its principles, namely, coordinating the various representations of conics through transformations by conversion, integrating their forms in the case of algebraic representations.

According to the evaluation of the participating teachers in the validation process, the prototype allows the visualization of various representations of conic curves, providing different representations and simultaneous changes, that is, dynamic articulations between visual variables and corresponding symbolic units. Additionally, it enables observing the curve as a section of a double cone based on elements of a curve plotted in the coordinate system on the Cartesian plane.

Overall, the prototype was characterized as a multirepresentational environment that facilitates dynamic coordination between synthetic and analytical geometry approaches. It proves effective in addressing difficulties related to coordinating different representations of conics and enabling transformations through treatment, allowing coordination between different algebraic expressions of the same conic curve.

When referring to a didactic resource for classroom use, the Conics Studium 3D prototype appears as an artifact capable of unifying the concept of conics and exploring the representation system of these curves. In this sense, it takes the place of other software, such as dynamic geometry software, often used to simulate various representations.

Therefore, the prototype fulfills its purposes, namely, unifying the concepts of conics and serving as a system capable of representing and coordinating their various representations simultaneously through transformations by conversion, and in the case of algebraic representations, integrating their forms through treatment.

Although we managed to meet most of the elicited requirements, it was not possible to implement the principle of direct manipulation, and some requirements, mostly associated with operational aspects such as input registration through any representation; demonstrating eccentricity through an algebraic expression; offering options to hide and show any of the geometric or algebraic objects; and providing direct manipulation of algebraic and geometric objects. We emphasize that offering options to hide and show any of the geometric or algebraic objects does not have a direct relationship with the categorized difficulties, but it can facilitate the execution of certain teaching situations.

In addition to the limitations mentioned earlier, some functionalities that were not initially targeted were suggested, such as providing tools for constructing conics by locus, similar to dynamic geometry software; incorporating a feature that allows observing the "unfolding of a curve into another through changes," for example, through visual variables or corresponding symbolic units, without the need for the selector; and introducing a ratio between two segments whose measure could be manipulated and reflected as the eccentricity of a curve described in the other windows.

If we consider that our study might provide elements for constructing a dynamic representation system to integrate and coordinate various representations of mathematical objects simultaneously, we are aware that the field of developing such computational artifacts, either with the help of resources used in the prototype's development or others, remains largely unexplored.

The results obtained at the end of our research can serve as a starting point, especially the Conics Studium 3D prototype and the new requirements highlighted earlier, in the search for an authoring system for teaching situations in a computational environment with the ability to integrate multiple artifacts, facilitating the manipulation and coordination of representation systems, particularly in the case of conics. It would function as an integrated and dynamic representation system, guiding the user through direct manipulation and observation to construct knowledge based on exploring the behaviors of objects and phenomena in a variety of records and definitions.

We will need to develop a new version on a platform independent of Rhino and the Grasshopper plugin, incorporating new artifacts, such as a dynamic geometry environment, and refining others.

This requires extensive research work, which we have only partially resolved, moving forward with theoretical and methodological choices that favored the development of studies, especially the results obtained in the theoretical phase, based on the four dimensions. However, we cannot overlook the strategy of using Rhino and the Grasshopper plugin, an adapted environment for the Conics Studium 3D artifact prototyping process. This approach can be understood as a simulation proposal capable of enabling the collection of experimental research data. We believe this issue deserves further exploration, including expansion to other mathematical objects.

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