ARITHMETIC WORD PROBLEMS AND ALGEBRA WORD PROBLEMS

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ABSTRACT

Algebra Word Problems are defined as Word Problems whose solution requires the use of the additive or multiplicative properties of equality. Arithmetic Word Problems, in contrast, do not require the stated properties for their solution, but only the substitution of equals. Evidence is given, showing that algebra problems are more difficult than arithmetic problems.

Keywords: Arithmetic Teaching; Word Problems; Levels of Difficulty in Word Problems.

RESUMO

Situações-Problema Algébricas são definidas como Situações-Problema cuja solução requer o uso das propriedades aditivas ou multiplicativas da igualdade. Situações-Problema Aritméticas, em contraste, não requerem as referidas propriedades para a sua solução, mas apenas a substituição de expressões iguais. Oferece-se evidência de que Situações-Problema Algébricas são mais difíceis que Situações-Problema Aritméticas.

Palavras-chave: Ensino de Aritmética; Situações-Problema; Níveis de Dificuldade em Situações-Problema.

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Two recurrent themes in research on word problems (see, for example Linchevski e Hercovics, 1996) are, first, the delineation of the relation between arithmetic and algebra word problems and, second, the determination of those factors that make these problems more, or less, difficult for the student. Taking a hint from the History of Mathematics, we suggest that these two kinds of problems can be distinguished by considering what properties of equality are used in solving them. Once this distinction is clearly drawn, we can hypothesize that, all things being equal, algebra word problems are more difficult than their arithmetical counterparts. We thus revisit an important study – Nesher, Greeno and Riley (1982) – that dealt with the relative difficulty of Additive Word Problems from a semantic point of view and show that the results of this study are easily explained on our hypothesis. We conclude by observing that our results can be useful to classroom teachers in that it helps in the appropriate sequencing of these kinds of problems.

THE DISTINCTION BETWEEN ARITHMETIC AND ALGEBRA PROBLEMS

Most of the basic definitions of "arithmetic" have to do with the computation of numbers, ofttimes including mention of the "four operations". Those of "algebra", in contrast, emphasize reasoning about quantitative relations through the use of "letters" and/or other symbols. Indeed, these definitions do seem to capture the essence of what is usually called arithmetic and (high school) algebra, as well as pointing to the major difference between the two. Thus, algebra is usually thought of as "an extension and a generalization of arithmetic. It provides tools for solving problems that arithmetic does not provide" (Leitzel, 1989, p. 26). In so doing, it adopts new conventions that are often in conflict with arithmetical conventions thereby causing difficulties for the learner.

Given this understanding of the situation, researchers like R. C. Lins and J. Gimenez (1997, p. 113) affirm that we need to understand better the relations between arithmetic and algebra in order to determine what they have in common, which will enable us to take a unified approach to teaching these two subjects. In a way, our proposal herein will be exactly the opposite from that of Lins and Gimemez, since we think that a better understanding of the relations between arithmetic and algebra will allow us to sort out arithmetic and algebra problems in a perspicacious way. This, in turn, will allow us to pay more appropriate attention to the reasoning involved in each of these types of problems, resulting in more effective teaching strategies. Ironically, a *prima facie* reason for adopting an alternative stance is exactly the commonality among virtually all branches of mathematics. That is, just as with John Fowles' *The Magus*[3], mathematics is a self-referential[4] enterprise, each part contributing a new understanding of other parts. Thus, what is needed is an operational way of determining when the computation of numbers is arithmetical and when it is algebraic.

Again, and perhaps more to the point of the present article, the history of mathematics shows that the use of “letters” is not essential to algebra. Indeed, all algebra done before François Viète's *Isagoge in Artem Analyticem* (*Introduction to Analytic Art*, 1591) was, in G. H. F. Nesselman’s terms[5], either rhetorical or syncopated. Helena M. Pycior (1997), for example, details how some important mathematicians actually rejected the symbolical approach – some of them on pedagogical grounds! Thus, it may come as not entirely surprising to find that rhetorical algebra, disguised as arithmetic, persists in the schools.

Our position will become clearer if we shift the focus of the discussion away from characterizing arithmetic and algebra as autonomous subject areas and concentrate on characterizing arithmetic and algebra problems. To do so, we ask what is characteristic about algebraic thinking in solving algebra problems. We already know that, since algebra can be done rhetorically, algebraic thinking does not necessarily involve symbolizing the argument. Once again history can be helpful. Hans Wussing (19, p. 82), for example, relates that the Arabian speaking mathematicians referred to algebra as *al-yabr* and *muqubala*, approximately "to put" and "to reduce". This is a fairly good description of what we do when we solve first

[3] For one man's view of this work, see Fossa (1989).
[4] The "self-referential" aspect does not necessarily imply paradoxical consequences. See Erickson and Fossa (1998).
[5] Nesselman made the distinctions in his *Versuch einer Kritischen Geschichte der Algebra*, published in 1842.

and second degree equations ("high school algebra") by isolating the unknown or rearranging an equation in order to factor it. Thus, it would seem that this kind of algebraic reasoning amounts to transforming equations into equivalent equations until the solution becomes evident or until a standard rule can be applied. We suggest that arithmetic problems, in contrast, are not solved by transforming equations into equivalent equations, but by substituting expressions for equivalent expressions by registering the result of performing one or more arithmetic operations. In the first case, that of algebra, the reasoning involves recognizing that performing the same operation on two equals will produce two ("other") equals. In the second case, that of arithmetic, only one operation is performed and the result is usually one of simplification.

At this point, some examples will be helpful. First we look at the following two examples of arithmetic problems:

$$\begin{aligned} x &= 3+(17+18) \\ &= 3+35 \\ &= 38 \end{aligned} \qquad \begin{aligned} x &= 3y+(17y+18\mathrm{y}) \\ &= 3y+35y \\ &= 38y \end{aligned}$$

In both of the above examples the equals sign is merely used to register the fact that the substitution of equals does not alter the result. Thus, 17+18 = 35 (or $17y+18y = 35y$, so, once again, we see that the use of "letters" does not serve to distinguish the arithmetical from the algebraic) and, hence, 3+(17+18) = 3+35. In structural terms, we use the substitution of equals and the transitivity of equality (to affirm that $x = 3+35$). The following two algebra problems are quite different from the problems given above:

$$\begin{gathered} x-17 = 21 \\ x-17+17 = 21+17 \\ x = 38 \end{gathered} \qquad \begin{gathered} 2x = 38 \\ \tfrac{1}{2}(2x) = \tfrac{1}{2}(38) \\ x = 19 \end{gathered}$$

In the first example, we add 17 to both sides of the equation and, thus transform the whole equation into a different, albeit equivalent, one. In fact the whole point of using the equals sign in the second line is to affirm that the transformation of the equation into an equivalent one has occurred. The reasoning here is vertical in the sense that we are claiming that the first line ($x-17 = 21$) and the second line ($x-17+17 = 21+17$) somehow express the same relation (they have the same solution).

Contrast this with the first arithmetic problem above, in which the reasoning, despite its vertical disposition on the page, is horizontal; that is, we could express it as

$$x = 3+(17+18) = 3+35 = 38.$$

The algebra problem cannot be so treated; that is, we cannot write x–17 = 21 = 21+17, nor even x–17+17 = x–17 = 21 = 21+17. The second algebra problem similarly uses vertical reasoning, except that instead of adding 17 to both sides of the equation, we multiplied both sides by ½.

Note that, in going from the second line to the third line of the algebra problems, we employed arithmetical, horizontal reasoning. In fact, we can, in contrast to the inference from the first to the second lines, express this move by expanding the second line both to the left and to the right in the following way:

$$x-17+17 = 21+17 \quad \Rightarrow \quad x = x-17+17 = 21+17 = 38$$

and

$$\tfrac{1}{2}(2x) = \tfrac{1}{2}(38) \quad \Rightarrow \quad x = \tfrac{1}{2}(2x) = \tfrac{1}{2}(38) = 19.$$

Thus, it appears that *one* important way in which algebra is an extension of arithmetic is that algebra employs both horizontal and vertical reasoning, whereas arithmetic only employs horizontal reasoning. In fact, this difference is so radical that it can be used to distinguish between arithmetic and (elementary) algebra.

To say that an arithmetic problem is one that can be solved using horizontal reasoning, whereas an algebra problem is one in which we need to use vertical reasoning, is a nice formulation of the result. Nonetheless, it is rather vague unless we can specify just what horizontal and vertical reasoning are. The discussion above, however, makes it clear as to how we can go about doing this in terms of mathematical structures. Before doing so, however, we note that, in the examples above, we left certain structural properties, such as the associative properties of addition and multiplication, implicit. That is because these are properties of the operations and will not come into play in our discussion. What is important for distinguishing between arithmetic and algebra problems are not the properties of the operations, but the properties of equality.

Now, among the most basic properties of equality, we may enumerate the following: those definitive of equivalence relations (reflection, symmetry and transitivity), substitutivity of equals, and the additive (equals added to equals are equal)[6] and multiplicative (equals multiplied by equals are equal) properties. The first two of these groups are characteristic of horizontal reasoning, whereas the additive and multiplicative properties of equality are characteristic of vertical reasoning. This is because vertical reasoning is needed in order (basically) to group numbers and variables (transpose them from one side of the equation to the other) and isolate variables; this is done, in the simplest cases[7], by the adding to both sides of the equation a suitably chosen additive inverse or multiplying by a suitably chosen multiplicative inverse. In the first algebra problem above, for example, we obtained x–17+17 = 21+17 from x–17 = 21 by adding 17 (the additive inverse of –17) to both sides of the latter equation; the net result was to transpose the number to the other side of the equation (changing its sign, naturally). In the second algebra problem, we multiplied by ½ (the multiplicative inverse of 2) in order to isolate the variable x on the left-hand side of the equation.

We can now define, in the proper context, an arithmetic problem as one that does not need to use, either implicitly or explicitly, the additive or multiplicative properties of equality for its solution. Similarly, we define, in the proper context, an algebra problem as one the requires, either implicitly or explicitly, the additive or multiplicative properties of equality for its solution. The qualifying phrase, "in the proper context", should be clear and, in any case will not be filled in here, since we are interested in applying this definition to word problems and, hence, will make the appropriate contextualization in that setting.

Before turning our attention to word problems, however, we mention the following practical criterion for distinguishing between arithmetic and algebra problems. If the variable is isolated on one side of the equation, the problem is arithmetical. Otherwise, it is algebraic.

[6] The natural language formulation in the text is actually very general, allowing a=b and c=d implies a+c=b+d. We will actually only need the case by which a=b implies a+c=b+c. Similar remarks apply to the natural language formulation of the multiplicative property.

[7] Clearly other cases abound, as in the process of "completing the square" to resolve quadratic equations. Nevertheless, the simplest cases (first degree equations in a single variable) may serve as our model without prejudicing the argument.

WORD PROBLEMS

Word problems are problems formulated in the natural language; mathematical word problems are word problems that are susceptible to formalization in some mathematical theory.[8] We now wish to restrict our attention to a certain type of mathematical word problem, which we will call Elementary Word Problems. These are mathematical word problems that can be formalized by a first degree equation in a single variable and can be solved using the four elementary arithmetic operations of addition, subtraction, multiplication and division. Building on the discussion from the previous section, we see that these problems can be divided into the following two types:

> *Arithmetic Problems* are those Elementary Word Problems that do not require, explicitly or implicitly, the use of the additive or multiplicative properties of equality for their solution.

> *Algebra Problems* are those Elementary Word Problems that require, explicitly or implicitly, the use of the additive or multiplicative properties of equality for their solution.

Clearly, any given problem either requires or does not require the use of the stated properties and, thus, there are no Elementary Word Problems that do not belong to one or the other of these two disjoint types.

The following is an example of an Arithmetic Problem:

[8] It is arguable that all word problems can be formalized mathematically. Perhaps we'd better say that mathematical word problems are those that are used as examples in mathematics classes! In any case, the remarks made in the paragraph in the text are not meant as definitions, but solely as an intuitive demarcation of the universe of discourse.

Ex. 1. A movie theater has 25 rows each of which contains 18 seats. Given that it is not allowed to stand or sit in the aisles, how many theatergoers does it take to sell out the theater three times?

The following, in contrast, is an example of an Algebra Problem:

Ex. 2. Three identical movie theaters have 25 rows each and each row has the same number of seats. If the total capacity of the three theaters, all together, is 1350 theatergoers, how many seats are there in each row?

In Ex. 1, we have $X = 3\times(25\times18)$ and it is sufficient to perform the two multiplications in order to reach the solution. In Ex. 2, however, we have $3\times(25\times X) = 1350$. To obtain the solution of this problem, we must transform the equation into an equivalent one, either by making two algebra "moves" to obtain $X = (1350\div3)\div25$, or by making an arithmetic move and an algebra move to obtain $X = 1350\div75$. It is possible for the student to arrive at the last equation ($X = 1350\div75$) as his/her mathematical model of the problem. This does not mean that the problem is an Arithmetic Problem for that student. It merely means that the multiplicative property was used implicitly, at the semantic, instead of the syntactic, level.

As the foregoing problems indicate, Elementary Word Problems may be composites of simpler Elementary Word Problems, which leads us to specify the following two types of problems:

Simple Problems are those Elementary Word Problems that require the use of a single operative move for their solution.

Combined Problems are those Elementary Word Problems that require the use of more than one operative move for their solution.

We use the rather inelegant phrase "operative move" instead of "operation" in order to emphasize that a problem, in which, for example, the single operation of multiplication is used twice, is to be classified as a Combined Problem. The two examples given above (Ex. 1 and Ex. 2) are both Combined Problems. We will refer to problems like the first of these as *Combined Arithmetic Problems* and to those like the second as *Combined Algebra Problems*. Likewise, we will refer to their simple counterparts as *Simple Arithmetic Problems* and *Simple Algebra Problems*.

There is one other type of Elementary Word Problem which deserves special attention. In this kind of problem, one or more of the unknown values is given in terms of another unknown (and, thus, a problem apparently containing various variables is reduced to one containing but a single variable). We describe this situation by saying that the unknowns are given recursively. Consequently, we have

> Structured *Problems* are those Elementary Word Problems in which at least one unknown is given recursively in terms of another unknown.

The following is an example of a Structured Problem:

> Ex. 3. Three friends divide \$210 so that the second gets \$50 more than the first and the third, \$80 more than the second. How much does each get?

Here the first of the three friends gets X dollars, the second, X+50 and the third, (X+50)+80. Thus, we have X+(X+50)+[(X+50)+80] = 210, so this is also a Combined Algebra Problem. It is theoretically possible to have Structured Arithmetic Problems; for this to occur the coefficient of the unknown would have to reduce, by using only arithmetic moves, to the number 1 *and* the constants on the side of the equation on which the unknown occurs would have to cancel out, again by using only arithmetic moves. We have not encountered any such problem in our research and so we conclude that, for all practical purposes, all Structured Problems are Combined Algebra Problems.[9] This being so, we will change our terminology slightly and call the

[9] In this regard, one could, if one wanted to get fancy, make the Arithmetic Problems a subset of the Algebra Problems by calling them Algebra Problems in which the only multiplicative inverse used is that of 1 and the only additive inverse used is that of 0. There seems, however, but little to be gained by doing so. Nevertheless, it might be of interest to present, at the appropriate time, the solution of Arithmetic Problems in a vertical format. That is, after the student has mastered X = 3×(25×18) = 3×450 = 1350, the teacher could rework the solution in the following way:

$$X = 3\times(25\times18)$$
$$X = 3\times450$$
$$X = 1350,$$

general class of Algebra Problems with two or more operative moves *Combined Algebra Problems in the Wide Sense*, thereby reserving the term *Combined Algebra Problems* for those in the Wide Sense that are not Structured Problems.

We can, thus, classify Elementary Word Problems according to the scheme given in Figure 1.

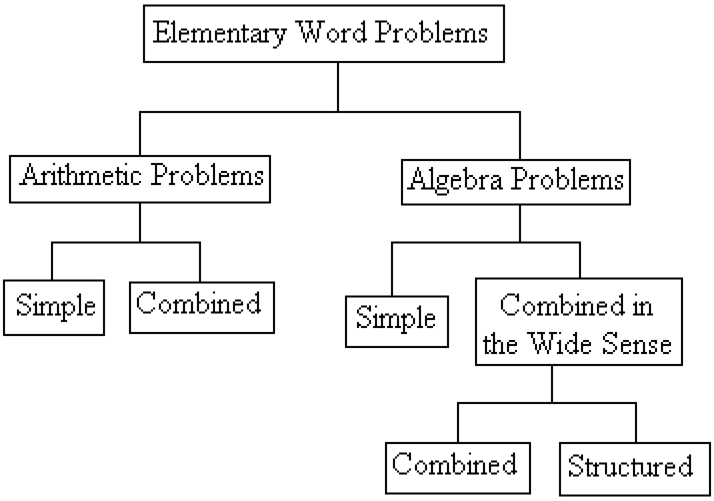

Figure 1. Classification of Word Problems.

SOME EXAMPLES

It will be instructive to look at a few more examples, since they will make evident certain contrasting features of Arithmetic and Algebra Problems. We may limit ourselves to Simple Problems, since the conclusions that we will draw will obviously carry over to the non-Simple types. To each example we will append the equation which is its mathematical model. We start with the following examples of Simple Arithmetic Problems:

Ex. 4.	I already had \$50 when I won \$20 at a raffle. How much did I have then?	$X = 50+20$
Ex. 5.	A salesman had 150 yards of wire. He sold 80 yards of it. How much wire did he have left after the sale?	$X = 150–80$
Ex. 6.	A theater has 15 rows, each of which contains 18	$X = 15\times18$

conceptualizing these as equivalent equations, thereby making a transition from Arithmetic to Algebra Problems. At the present time, however, we do not have any data pertinent to the efficacy of this procedure. Observe that by leaving out the X in the second and third lines above, as is often done, makes it harder to conceptualize these lines as separate (though equivalent) equations.

	seats. How many seats does the theater have in all?	
Ex. 7.	Jane needs to distribute 1200 lollipops evenly into 5 boxes. How many lollipops should she put in each box?	$X = 1200 \div 5$

In contrast, we have the following examples of Simple Algebra Problems:

Ex. 8.	After putting $25 dollars in her safe, my sister had saved $78. How much did she have before making this deposit?	$X+25 = 78$
Ex. 9.	My brother gave me some money to spend at the supermarket. I spent $156 and still had $95 left. How much money did my brother give me?	$X - 156 = 95$
Ex. 10.	My boss had 200 yards of wire in his shop. After one of the other salesmen sold some of it, I found that there was only 189 yards left. How much wire did that other salesman sell?	$200 - X = 189$
Ex. 11.	The triple of a certain number is 120. What is the number?	$3 \times X = 120$
Ex. 12.	Joy gave out 28 toys to the children at her birthday party. If each child got 4 toys, how many children were at Joy's party?	$28 \div X = 4$
Ex. 13.	At Sarah's birthday party there were 9 children, each of whom received 5 toys. How many toys were given out at Sarah's party?	$X \div 9 = 5$

We may also generalize the mathematical models of these examples in the following way:

Simple Arithmetic Problems	Simple Algebra Problems
$X = a+b$	$X+a = b$
$X = a-b$	$X-a = b$
	$a-X = b$
$X = a \times b$	$a \times X = b$
$X = a \div b$	$a \div X = b$
	$X \div a = b$

From these examples it becomes evident that the equations corresponding to Arithmetic Problems have the variable isolated on one side of the equation, thereby making it possible to solve the equation by simply performing the indicated operation (or operations, in the case of Combined Problems) in a horizontal manner. The equations corresponding to Algebra Problems, in contrast, do not have the variable isolated on one side of the equation. Consequently, in order to isolate it, the student must use the inverse operation in such a way as to transform the equation into an equivalent equation: to so he/she has to use vertical reasoning.

In horizontal reasoning, the student need only see the equation as a process which registers, at each step, the result of the operation performed. In vertical reasoning, however, the equation itself must be reified[10] so that it can become the object of thought. Thus, vertical reasoning is a more sophisticated way of thinking than horizontal reasoning, which leads us to propose the following thesis: *Algebra (Word) Problems are generally more difficult than Arithmetic (Word) Problems*.

In what follows we will partially address this thesis by reviewing a classic study that takes into account relative difficulty of Additive Word Problems, that is, problems that only involve the operations of addition and/or subtraction.[11]

SEMANTIC CATEGORIES

Nesher, Greeno and Riley (1982) proposed that Additive Word Problems can be grouped according to the following semantic categories:

- Combine Problems: those which emphasize static relations among quantities
- Change Problems: those which emphasize an increase or decrease from an initial state to a final state.
- Compare Problems: those which emphasize a comparison of quantities.

Combine problems are conceived of relating a total (t) to the sum or the difference of its parts (p and q) and can thus be represented by the equation $t = p \pm q$. In Change Problems, we have a final state (f) as the result of an initial state (i) plus or minus some change (c), or $f = i \pm c$. Finally, in Compare Problems, we have a

[10] See, for example, Sfard (1992).

[11] Our thesis is consonant with Hiebert (1982), who found that the position of the variable influences the level of difficulty of the problem.

difference (d) between a greater quantity (g) and a lesser quantity (l), or $d = g–l$. One of these two quantities, the greater or the lesser, is the standard (s) of the comparison to which the other (c) is compared. We may indicate this by subscripts in the equation when these notions come into play, giving us $d = g_{s,c}–l_{c,s}$.

Each of these categories can be further subdivided in various ways and the difficulty of each category is related to the subdivisions. We present the subdivisions in the following table, which also includes the mathematical equation that models each type, along with its characterization as an Arithmetic Problem or an Algebra Problem, according to the definitions presented above.

Type	Description	Equation	Arithmetic/ Algebra
Combine Problems $t = p \pm q$			
1.	Asks for the total.	$X = p \pm q$	Arithmetic
2.	Asks for a part.	$t = X \pm q$	Algebra
Change Problems $f = i \pm c$			
1.	Asks for final state of an increase.	$X = i+c$	Arithmetic
2.	Asks for final state of a decrease.	$X = i–c$	Arithmetic
3.	Asks for amount of increase.	$f = X+c$	Algebra
4.	Asks for amount of decrease.	$f = X–c$	Algebra
5.	Asks for initial state of an increase.	$f = X+c$	Algebra
6.	Asks for initial state of a decrease.	$f = X–c$	Algebra
Compare Problems $d = g–l$ or $d = g_{s,c}–l_{c,s}$			
1.	Asks for how much the greater is more than the lesser.	$X = g–l$	Arithmetic
2.	Asks for how much the lesser is less than the greater.	$X = g–l$	Arithmetic
3.	Asks for the lesser as the compared quantity.	$d = g_s–X_c$	Algebra
4.	Asks for the greater as the compared quantity.	$d = X_c–l_s$	Algebra
5.	Asks for the lesser as the standard quantity.	$d = g_c–X_s$	Algebra
6.	Asks for the greater as the standard quantity.	$d = X_s–l_c$	Algebra

Table I. Semantic Categories.

Nesher, Greeno and Riley (1982) found that

- among the Combine Problems, those that ask for a part are more difficult;

- among the Change Problems, those that ask for the initial state are the most difficult;
- among the Compare Problems, those that ask for the standard are the most difficult.

By referring to the table, we find that all the problems that the study found to be most difficult are Algebra Problems.

The data also seem to imply that there are differences in difficulty among the Algebra Problems themselves. This, however, is not surprising. We understand these differences to be the result of the semantic considerations that come into play within the Algebra classification.

CONCLUSION

In the foregoing we presented a systematic and detailed analysis of the classical article Nesher, Greeno and Riley (1982) in terms our determination of the structural difference between arithmetic and algebra word problems. The analysis can be extended to other studies, for example: Rosenthal e Resnick (1974), Fayol and Abdi (1986), Fayol, Abdi and Gombert (1987), Hershkovitz Nesher and Novotná (2000), Hershkovitz, and Nesher (2003), Thevenot and Oakhill (2005), Elia, Gagatsis and Demetriou (2007), Thevenot, Devidal, Barrouillet and Feyol (2007), Swanson, Jerman and Zheng (2008) and Ilany and Margolin (2010). In some of these, we can see that other aspects are emphasized. Indeed, Kieran (2006) shows that three groups of topics have emerged since the mid-70s. They include the transition from arithmetic to algebra (emergent in the mid-70s), the use of computers and multiple representations (emergent in the mid-80s) and dynamic modeling of physical situations (emergent in the mid-90s). All of these topics are important in the investigation of students' performance on word problems. Nevertheless, since the structural factors that discriminate between Arithmetic Problems and Algebra Problems have not been heretofore identified and linked to levels of difficulty, we believe that many teachers unwittingly set inappropriate Word Problems for their students. The recognition of this factor thus may help teachers to sequence educational activities in a more appropriate manner, thereby contributing to greater effectiveness in students learning of how to deal with Mathematical Word Problems.

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